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FACTORIZATION IN THE GROUP ALGEBRA OF THE REAL LINE

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The algebra in question is the set L of all complex Lebesgue integrable functions on the real line, with pointwise addition, and with convolution as multiplication:

$$(g * h)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x - t)\, h(t)\, dt. \tag{1}$$

The norm

$$\|f\| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\, dx \tag{2}$$

makes L into a Banach algebra. The Fourier transform of a function $f \in L$ will be denoted by $\hat{f}$:

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ixy}\, dx. \tag{3}$$

Then $f = g * h$ if and only if $\hat{f}(y) = \hat{g}(y)\, \hat{h}(y)$ for all real y.

In this note it is proved that every member of L is the convolution of two others. Thus, although the algebra L has no unit element, unrestricted factorization is possible. There are no primes in L.

THEOREM. *Suppose that $f \in L$. There exist functions $g \in L$ and $h \in L$ such that*

(*a*) $f = g * h$;

(*b*) *both h and $\hat{h}$ are positive and even;*

(*c*) *g lies in the closed ideal generated by f.*

Proof: For $t > 0$, let K_t be the function whose Fourier transform is

$$\hat{K}_t(y) = \begin{cases} 1 - \dfrac{|y|}{t} & \text{if } |y| < t \\ 0 & \text{if } |y| \geqslant t. \end{cases}$$

Put $\sigma_t = f * K_t$, and $\alpha(t) = \|f - \sigma_t\|$. It is well known that $\alpha(t) \to 0$ as $t \to \infty$.

Choose a sequence $\{t_n\}$ as follows: $t_1 = 0$; for $n \geqslant 2$, $t_n > 2t_{n-1}$ and $\alpha(t) < n^{-2}$ if $t \geqslant t_n$. Construct a function ϕ, concave and with continuous second derivative in $[0, \infty)$, such that $\phi(t_n) = n$. Consideration of the graph of ϕ shows that $t_n \phi'(t_n) < 2$ for $n \geqslant 2$. Hence

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \alpha(t)\, t\, |\phi''(t)|\, dt &< -n^{-2} \int_{t_n}^{t_{n+1}} t\, \phi''(t)\, dt \\ &= n^{-2}\{t_n\, \phi'(t_n) - t_{n+1}\, \phi'(t_{n+1}) + \phi(t_{n+1}) - \phi(t_n)\} \\ &< 3n^{-2}. \end{aligned}$$

Consequently,

$$\int_0^{\infty} \alpha(t)\, t\, |\phi''(t)|\, di < \infty. \tag{4}$$

Now define

$$g(x) = f(x) + \int_0^\infty \{\sigma_t(x) - f(x)\}\, t\,\phi''(t)\, dt. \tag{5}$$

By relation (4) and the Fubini theorem, the integral in equation (5) converges absolutely for almost all x, and $g \in L$. Also,

$$\hat{g}(y) = \hat{f}(y) + \hat{f}(y) \int_0^\infty \{\hat{K}_t(y) - 1\}\, t\,\phi''(t)\, dt.$$

The last integral is an even function of y. For $y > 0$, it is equal to

$$-\int_0^y t\,\phi''(t)\, dt - y\int_y^\infty \phi''(t)\, dt = \phi(y) - \phi(0) = \phi(y) - 1.$$

Thus, if we put $\phi(-t) = \phi(t)$, we have, for all real y,

$$\hat{g}(y) = \hat{f}(y)\,\phi(y). \tag{6}$$

Next, put $\lambda(t) = 1/\phi(t)$, and

$$h(x) = \int_0^\infty K_t(x)\, t\,\lambda''(t)\, dt. \tag{7}$$

Note that λ is convex in $(0, \infty)$, that $\lambda(t) \to 0$ as $t \to \infty$, and that consequently

$$\int_0^\infty t\,\lambda''(t)\, dt = \lambda(0) < \infty.$$

This implies that $h \in L$, and

$$\hat{h}(y) = \int_0^\infty \hat{K}_t(y)\, t\,\lambda''(t)\, dt.$$

For $y > 0$, a calculation similar to the one that led to equation (6) shows that $\hat{h}(y) = \lambda(y)$. By equation (6),

$$\hat{f}(y) = \hat{g}(y)\,\hat{h}(y) \tag{8}$$

for all real y, and part (a) of the theorem is proved.

It is evident from the construction that part (b) holds. To prove part (c), note that the function $\phi\hat{K}_t$ satisfies a Lipschitz condition of order 1 and vanishes outside a bounded interval. Hence $\phi\hat{K}_t = \hat{U}_t$ for some $U_t \in L$, and, by equation (6), $\hat{U}_t\hat{f} = \hat{K}_t\hat{g}$. It follows that $K_t * g$ belongs to the ideal generated by f, for each $t > 0$. As $t \to \infty$, $K_t * g$ tends to g, in the norm of L, so that part c holds. This completes the proof.

It is quite natural to ask now whether every non-negative $f \in L$ is the convolution of two nonnegative members of L. To see that this is not the case, observe that the integral in equation (1) is a lower semicontinuous function of x if g and h are nonnegative, so that f must coincide almost everywhere with some lower semicontinuous function if $f = g * h$. But this is impossible if f is the characteristic function of a compact totally disconnected set of positive measure, for instance.

It would be interesting to determine those functions which are convolutions of nonnegative functions.

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