# COUNTING THE POSITIVE RATIONALS: A BRIEF SURVEY 

DAVID M. BRADLEY


#### Abstract

We discuss some examples that illustrate the countability of the positive rational numbers and related sets. Techniques include radix representations, Gödel numbering, the fundamental theorem of arithmetic, continued fractions, Egyptian fractions, and the sequence of ratios of successive hyperbinary representation numbers.


Ginsberg [7] gave an injective mapping of the rational numbers $\mathbf{Q}$ into the positive integers $\mathbf{Z}^{+}$by interpreting the fraction $-a / b$ as a positive integer written in base 12 , with the division slash and the minus sign as symbols for 10 and 11 , respectively. This idea was previously publicized by Campbell [2], who encountered it in the early 1970s, and also used it to establish the countability of the ring $\mathbf{Q}[x]$ of polynomials with rational coefficients. Citing Campbell and using essentially the same arguments, Touhey [21] gave injections of $\mathbf{Q}$ and $\mathbf{Z}[x]$ into $\mathbf{Z}^{+}$. More recently, Kantrowitz [14] formulated the main idea as a general principle, namely that the set of all words of finite length that may be formed from the letters of a countable alphabet is countable.

If we identify the set $\mathbf{Q}^{+}$of positive rational numbers with the subset of $\mathbf{Z}^{+} \times \mathbf{Z}^{+}$ consisting of the ordered pairs of coprime positive integers, then any injective mapping $\varphi: \mathbf{Z}^{+} \times \mathbf{Z}^{+} \rightarrow \mathbf{Z}^{+}$restricts to an injection of $\mathbf{Q}^{+}$into $\mathbf{Z}^{+}$. Additional examples follow. All maps $\varphi$ have domain $\mathbf{Z}^{+} \times \mathbf{Z}^{+}$and codomain $\mathbf{Z}^{+}$.

1. An extremely simple bijection is given by $\varphi(n, m)=(2 n-1) 2^{m-1}$. Bijectivity of $\varphi$ is equivalent to the fact that every positive integer has a unique representation as the product of an odd positive integer and a power of 2 .
2. Let $\varphi(n, m)=\left(2^{n}-1\right) 2^{m}$. It is easy to verify that $\varphi$ is injective using elementary divisibility properties of the positive integers. However, if $\left(2^{n}-1\right) 2^{m}$ is written in binary (base 2), then $\varphi$ can be regarded as the map which encodes the ordered pair ( $n, m$ ) as the string $\{1\}^{n}\{0\}^{m}$ of $n$ consecutive ones followed by $m$ consecutive zeros. From this viewpoint, injectivity is obvious.

Grant and Priest [9] offered the surjection $g: \mathbf{Z}^{+} \rightarrow \mathbf{Q}^{+}$defined by $g(x)=(k+$ $1) /(m+1)$, where $k$ and $m$ are the number of fours and sevens, respectively, in the

[^0]decimal representation of $x$. This is not entirely unrelated to our example, as changing the words "four" to "one", "seven" to "zero" and "decimal" to "binary" reveals.
3. Let $\varphi(n, m)=2^{n+m+1}-1-2^{m}$. Again, it is easy to verify directly that $\varphi$ is injective, and again there is a nice interpretation of the map via base change. Essentially, $\varphi$ encodes the ordered pair ( $n, m$ ) by writing $n$ and $m$ in unary (base 1 ) using the symbol 1 , separating the two strings of ones with a zero, and then interpreting the resulting string $\{1\}^{n} 0\{1\}^{m}$ as the binary (base 2) representation of the positive integer $2^{n+m+1}-1-2^{m}$ in the usual way.
4. A simpler injection from the viewpoint of the algebraic formulation of the rule is $\varphi(n, m)=2^{n}+2^{n+m}$, which translates to the binary string $1\{0\}^{m-1} 1\{0\}^{n}$.
5. Let $\varphi(n, m)=2^{n} 3^{m}$. Injectivity is a consequence of the fundamental theorem of arithmetic. This is a very simple example of what is sometimes referred to as a Gödel numbering, in which various elements are mapped to products of prime powers. The technique can be easily adapted to establish countability of a wide variety of countable sets, such as the set of all finite subsets of a countable set, a countable union of countable sets, and so on. In [5] it is used to establish countability of the ring $\mathbf{Q}[x]$ of polynomials with rational coefficients, an approach that contrasts with Campell's [2].
6. Let $\mathcal{P}=\{2,3,5,7,11, \ldots\}$ be the set of prime positive integers and let $p: \mathbf{Z}^{+} \rightarrow \mathcal{P}$ be injective. Then $\phi(n, m)=(p(m))^{n}$ defines an injection of $\mathbf{Z}^{+} \times \mathbf{Z}^{+}$into $\mathbf{Z}^{+}$. The related map $(p(m))^{n} \mapsto n / m$ is extended to a surjection of $\mathbf{Z}^{+}$onto $\mathbf{Q}^{+}$in [9].

The injections of examples 2 through 6 grow too quickly to be surjective. However, if the growth rate is reduced from exponential to quadratic, it is possible to get bijective maps distinct from the bijection of example 1 .
7. Let $\varphi(n, m)=(n+m-1)(n+m-2) / 2+n$. It is an interesting exercise to prove algebraically that $\varphi$ is bijective. By following Cantor, a visual proof is readily obtained. Regard $\mathbf{Z}^{+} \times \mathbf{Z}^{+}$as an infinite-dimensional matrix with $(n, m)$ in row $n$ and column $m$. In light of the identity $1+2+\cdots+(n+m-2)=\varphi(n, m)-n$, one sees that $\varphi$ lists the entries in order starting with $(1,1),(1,2),(2,1),(1,3),(2,2),(3,1)$ and traversing successive diagonals with $n+m$ constant so that as each diagonal is traversed, the row index increases as the column index decreases.
8. Picture $\mathbf{Z}^{+} \times \mathbf{Z}^{+}$as the lattice of points with positive integer coordinates in the first quadrant of the Cartesian plane. Instead of traversing diagonals, we can exhaust the lattice by tracing out successively larger upside-down capital L's, starting with $(1,1)$, then $(2,1),(2,2),(1,2)$, then $(3,1),(3,2),(3,3),(2,3),(1,3)$, and so on. In this scheme, the lattice point $(n, m)$ occurs at position $\varphi(n, m)=(\max (n, m))^{2}-\max (n, m)+m-n+1$ in the sequence. Again, it is an interesting exercise to prove algebraically that $\varphi$ is bijective.

Alternatively, one can start by listing the pairs $(n, m)$ with $n \leq m$ (the horizontal portion of the upside-down L's), and then insert into every other position the remaining
pairs obtained by switching $n$ and $m$ when they differ. This gives a list that begins

$$
\begin{equation*}
(1,1) ;(1,2),(2,1),(2,2) ;(1,3),(3,1),(2,3),(3,2),(3,3) ; \tag{1}
\end{equation*}
$$

and so on.
MacHale [16] gave a visual bijective correspondence between the complete integer lattice $\mathbf{Z} \times \mathbf{Z}$ and the positive integers using a spiral path starting at the origin. With a little effort, one should be able to provide an algebraic formula for the position of a generic lattice point in the sequence under this scheme as well. Other more complicated "arraybased" enumerations may be found in [8, 11, 13].
9. An explicit bijective correspondence between $\mathbf{Q}^{+}$and $\mathbf{Z}^{+}$can be obtained by exploiting the multiplicative structure of these sets more fully. Again, let $\mathcal{P}=\{2,3,5,7,11, \ldots\}$ be the set of prime positive integers. By the fundamental theorem of arithmetic, the map

$$
\begin{equation*}
\prod_{p \in \mathcal{P}} p^{\alpha_{p}} \mapsto \tilde{\alpha}=\left(\alpha_{2}, \alpha_{3}, \alpha_{5}, \alpha_{7}, \ldots\right) \tag{2}
\end{equation*}
$$

defines a bijection from $\mathbf{Q}^{+}$to the set $\mathbf{Z}^{*}$ of integer sequences indexed by $\mathcal{P}$, all but finitely many terms of which are zero. Let $\mathbf{N}$ denote the set of non-negative integers, and let $\mathbf{N}^{*}$ denote the set of all sequences of non-negative integers indexed by $\mathcal{P}$, all but finitely many terms of which are zero. If $\tilde{\alpha} \mapsto \tilde{\beta}$ is any bijection from $\mathbf{Z}^{*}$ to $\mathbf{N}^{*}$, then

$$
\begin{equation*}
\prod_{p \in \mathcal{P}} p^{\alpha_{p}} \mapsto \prod_{p \in \mathcal{P}} p^{\beta_{p}} \tag{3}
\end{equation*}
$$

defines a bijection from $\mathbf{Q}^{+}$to $\mathbf{Z}^{+}$.
For an example of a bijection $\tilde{\alpha} \mapsto \tilde{\beta}$, one could take for each $p \in \mathcal{P}$ any bijection $g_{p}: \mathbf{Z} \rightarrow \mathbf{N}$ such that $g_{p}(0)=0$, and set $\beta_{p}=g_{p}\left(\alpha_{p}\right)$. But it's probably simplest to use the same bijection $g_{p}=g: \mathbf{Z} \rightarrow \mathbf{N}$ for each coordinate. One possible choice for $g$ is the map

$$
\sum_{k=0}^{\infty} a_{k}(-2)^{k} \mapsto \sum_{k=0}^{\infty} a_{k} 2^{k}
$$

where each $a_{k} \in\{0,1\}$ and all but finitely many are zero. Another possibility is to let $g(\alpha)=2 \alpha$ if $\alpha \geq 0$ and $g(\alpha)=-2 \alpha-1$ if $\alpha<0$. With this choice, if for each $p \in \mathcal{P}$, $\beta_{p}=g\left(\alpha_{p}\right)$ in (3), then $3 / 5$ maps to $3^{2} 5=45$ and the two millionth positive rational number is $2^{-4} 5^{3}=125 / 16$. This is precisely the map presented in [19]; earlier we find the inverse map given in [6], but the idea of a bijection of the form (3) goes back earlier still-at least to [17]. In each instance after [17] the authors, journal editors and referees all seem to have been unaware that the same idea appeared previously.

Niven's bijective correspondence [18] between the rationals and the positive integers exploits the representation for the positive rationals as products of primes with integer exponents, as shown here on the left hand side of (3). In light of this and the fact that
he was a number theorist, it seems remarkable that he failed to exploit the corresponding (and if anything, more familiar) canonical representation of positive integers as products of prime powers, shown here on the right hand side of (3). Instead, he set up an intermediate correspondence between integers written in binary notation and sequences derived from examining consecutive blocks of ones in the binary representation. Thus, his bijection is unnecessarily complicated.
10. Let $p_{0}, p_{1}, p_{2}, \ldots$ be any enumeration of the prime positive integers (i.e. the map $n \mapsto p_{n}$ defines a bijection from $\mathbf{N}$ to $\mathcal{P}$ ). If we index the elements of $\mathbf{Z}^{*}$ by $\mathbf{N}$ instead of $\mathcal{P}$, then the map (2) becomes

$$
\prod_{n \in \mathbf{N}} p_{n}^{\alpha_{n}} \mapsto \tilde{\alpha}=\left(\alpha_{0}, \alpha_{1}, \alpha_{2}, \ldots\right)
$$

Let $x$ be an indeterminate. Forming the generating function of the sequence $\tilde{\alpha}$ gives a function from $\mathbf{Q}^{+}$to $\mathbf{Z}[x]$ defined by

$$
\prod_{n \in \mathbf{N}} p_{n}^{\alpha_{n}} \mapsto \sum_{n \in \mathbf{N}} \alpha_{n} x^{n}
$$

which is not merely a bijective set map, but in fact a group isomorphism [12, II.1, p. 75, ex. 11] between the multiplicative free abelian group $\left(\mathbf{Q}^{+}, \cdot\right)$ with basis $\mathcal{P}$ and the additive free abelian group $(\mathbf{Z}[x],+)$ with basis equal to the non-negative integer powers of $x$. The abelian monoids $\left(\mathbf{Z}^{+}, \cdot\right)$ and $(\mathbf{N}[x],+)$ are likewise isomorphic. Thus, given any bijection $\sum_{n \geq 0} \alpha_{n} x^{n} \mapsto \sum_{n \geq 0} \beta_{n} x^{n}$ from $\mathbf{Z}[x]$ to $\mathbf{N}[x]$, we get a bijection from $\mathbf{Q}^{+}$to $\mathbf{Z}^{+}$via

$$
\prod_{n \in \mathbf{N}} p_{n}^{\alpha_{n}} \mapsto \sum_{n \in \mathbf{N}} \alpha_{n} x^{n} \mapsto \sum_{n \in \mathbf{N}} \beta_{n} x^{n} \mapsto \prod_{n \in \mathbf{N}} p_{n}^{\beta_{n}}
$$

11. The theory of continued fractions yields additional possibilities. It is known 10 that every non-negative rational number $r$ has a unique representation as a finite continued fraction of the form

$$
r=a_{0}+\frac{1}{a_{1}+\frac{1}{a_{2}+\frac{1}{a_{n-1}+\frac{1}{a_{n}+\frac{1}{1}}}}},
$$

where $a_{0}, a_{1}, \ldots, a_{n}$ are integers with $a_{0} \geq 0$ and $a_{j} \geq 1$ for $1 \leq j \leq n$. Therefore, if $b_{k}=\sum_{j=0}^{k} a_{j}$ for $0 \leq k \leq n$, then

$$
r \mapsto \sum_{k=0}^{n} 2^{b_{k}}
$$

defines a bijection from the set $\mathbf{Q}_{\geq 0}$ of non-negative rational numbers to the set $\mathbf{Z}^{+}$of positive integers. A more complicated bijective correspondence between $\mathbf{Q}_{\geq 0}$ and $\mathbf{Z}^{+}$ using continued fractions is described in [4].
12. Of course, a bijection from $\mathbf{Z}^{+}$to $\mathbf{Q}^{+}$may viewed as a sequence which enumerates each positive rational exactly once. The recursion [20]

$$
\begin{equation*}
\gamma_{1}=1, \quad \gamma_{2 k}=1+\gamma_{k}, \quad \gamma_{2 k+1}=1 / \gamma_{2 k}, \quad k \in \mathbf{Z}^{+} \tag{4}
\end{equation*}
$$

defines one such sequence. We briefly reproduce the motivation for this definition here. Start by enumerating the finite non-empty sets of positive integers: let $M_{1}=\{1\}$ and for $k \geq 1$, let $M_{2 k}=\left\{n+1: n \in M_{k}\right\}, M_{2 k+1}=M_{2 k} \cup\{1\}$. An easy induction argument proves that for each positive integer $k$, $\left\{M_{j}: 1 \leq j<2^{k}\right\}$ coincides with the collection of non-empty subsets of $\{1,2, \ldots, k\}$. Next, define an enumeration of all finite sequences of positive integers in such a way that $S_{k}=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ if and only if $M_{k}=\left\{a_{1}, a_{1}+a_{2}, \ldots, a_{1}+a_{2}+\cdots+a_{n}\right\}$. Finally, for $k \in \mathbf{Z}^{+}$, let

$$
\rho_{k}=\frac{1}{a_{1}+\frac{1}{a_{2}+\frac{1}{\ddots++\frac{1}{a_{n}+\frac{1}{1}}}}}
$$

and $\gamma_{k}=1 / \rho_{k}-1$. Then $\gamma_{1}=1$, and if $n>1$, then

$$
\gamma_{k}=a_{1}-1+\frac{1}{a_{2}+\frac{1}{a_{3}+\frac{1}{\ddots \cdot+\frac{1}{a_{n}+\frac{1}{1}}}}} .
$$

Since $S_{2 k}$ is just $S_{k}$ with $a_{1}$ replaced by $a_{1}+1$ and $S_{2 k+1}$ is just $S_{k}$ with an extra 1 in front, the recurrence (4) follows immediately. The fact that every positive rational occurs precisely once in the sequence $\gamma_{1}, \gamma_{2}, \gamma_{3}, \ldots$ follows from the existence and uniqueness of the continued fraction representation.
13. Lauwerier [15, p. 23] describes a listing of the positive rationals between 0 and 1 arranged first by increasing denominator and then by increasing numerator:

$$
\frac{1}{1}, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{2}{3}, \quad \frac{1}{4}, \quad \frac{3}{4}, \quad \frac{1}{5}, \quad \frac{2}{5}, \quad \frac{3}{5}, \quad \frac{4}{5}, \quad \frac{1}{6}, \quad \frac{5}{6}, \ldots
$$

A complete list of the positive rationals is obtained by inserting the reciprocals:

$$
\frac{1}{1}, \quad \frac{1}{2}, \quad \frac{2}{1}, \quad \frac{1}{3}, \quad \frac{3}{1}, \quad \frac{2}{3}, \quad \frac{3}{2}, \quad \frac{1}{4}, \quad \frac{4}{1}, \quad \frac{3}{4}, \quad \frac{4}{3}, \quad \frac{1}{5}, \quad \frac{5}{1}, \quad \ldots
$$

This list can also be obtained from (11) by sending the ordered pair $(n, m)$ to $n / m$, omitting pairs in which $n$ and $m$ have a common divisor greater than 1 .
14. An enumeration of the positive rationals with combinatorial significance is discussed in [1]. Let $b_{0}=1$ and for $n \in \mathbf{Z}^{+}$, let $b_{n}$ be the number of hyperbinary representations of $n$. That is, $b_{n}$ is the number of ways to write $n$ as a sum of powers of 2 , each power being used at most twice. For example, $b_{5}=2$ because $5=2^{2}+2^{0}=2^{1}+2^{1}+2^{0}$. It's not hard to see that for all $k \geq 0, b_{2 k+1}=b_{k}$ and $b_{2 k+2}=b_{k}+b_{k+1}$. This recursion together with the initial condition gives an alternative way of defining the sequence of hyperbinary representation numbers; it also provides a convenient way to compute the first several terms. Calkin and Wilf proved that each positive rational number occurs once and only once in the list $b_{0} / b_{1}, b_{1} / b_{2}, b_{2} / b_{3}, \ldots$ Thus, the map $\psi: \mathbf{Z}^{+} \rightarrow \mathbf{Q}^{+}$defined by $\psi(k)=b_{k-1} / b_{k}$ is bijective.
15. Cohen [3] proves inter alia that every positive rational $r$ such that $0<r<1$ has a unique representation as a sum of unit fractions of the form

$$
r=\sum_{j=1}^{k} \prod_{i=1}^{j} \frac{1}{n_{i}}
$$

where the $n_{i}$ are integers and $2 \leq n_{1} \leq n_{2} \leq \cdots \leq n_{k}$. If we let $\psi(r)=\sum_{j=1}^{k} 2^{n_{j}+j-3}$, then the map $x \mapsto \psi(x /(x+1))$ defines a bijection from $\mathbf{Q}^{+}$to $\mathbf{Z}^{+}$.

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Department of Mathematics \& Statistics, University of Maine, 5752 Neville Hall Orono, Maine 04469-5752, U.S.A.

E-mail address: bradley@math.umaine.edu, dbradley@member.ams.org


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