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Location of the Zeros of Polynomials

Author(s): Q. G. Mohammad

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As before, since  $F(0) \leq K(0)$ , there is a point  $X \in [0, 1]$  such that  $F(X) = K(X)$ , and the proof is completed.

It might be noted that if  $(p) + S(a, b) \subset E_2$ , Theorem 1 is valid for the multiplier  $1/\sqrt{3}$  (which is *less* than 1). In general, however, 1 is the smallest multiplier of  $pf$  for which Theorem 1 is valid. This can be seen by considering the Minkowski space  $M_2^{(1)}$ . The elements of this space are ordered pairs of real numbers with distance of  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$  taken as

$$xy = |x_1 - y_1| + |x_2 - y_2|.$$

Let  $S(a, b)$  be contained in the interval joining  $(-1, 0)$  and  $(1, 0)$  and let  $p = (0, 1)$ . It is easily seen that if  $p, x, y$  form an equilateral triple with  $x, y \in S(a, b)$  then  $x$  and  $y$  *must* be the points  $(-1, 0)$  and  $(1, 0)$ . Hence  $S(a, b)$  is the interval joining these two points; the foot  $f$  of  $p$  on  $S(a, b)$  is the origin, and  $\min[af, bf] = pf$ .

A slightly more complicated example can be given, also in the  $M_2^{(1)}$ , which shows that 2 is the smallest multiplier of  $pf$  for which Theorem 2 is valid.

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#### References

1. L. M. Blumenthal, *Theory and applications of distance geometry*, Clarendon Press, Oxford, 1953.
2. L. M. Blumenthal and C. V. Robinson, A new characterization of the straight line, *Reports of a Mathematical Colloquium, Second Series*, 2 (1940) 25-27.

#### LOCATION OF THE ZEROS OF POLYNOMIALS

Q. G. MOHAMMAD, Jammu and Kashmir University, India

The following theorem is due to Montel and Marty [1, p. 107].

THEOREM A. *All the zeros of the polynomial*

$$p(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + z^n$$

*lie in  $|z| \leq \max(L, L^{1/n})$  where  $L$  is the length of the polygonal line joining in succession the points  $0, a_0, a_1, \cdots, a_{n-1}, 1$ ; i.e.*

$$L = |a_0| + |a_1 - a_0| + \cdots + |a_{n-1} - a_{n-2}| + |1 - a_{n-1}|.$$

We prove

THEOREM 1. *All the zeros of the polynomial of Theorem A lie in  $|z| \leq R = \max(L_p, L_p^{1/n})$  where*

$$L_p = n^{1/q} \left( \sum_{i=0}^{n-1} |a_i|^p \right)^{1/p}, \quad p^{-1} + q^{-1} = 1.$$

*The bound is sharp.*

*Proof.* We have

$$(1) \quad |p(z)| \geq |z|^n \left( 1 - \sum_{i=1}^n \frac{|a_{n-i}|}{|z|^i} \right) \geq |z|^n \left\{ 1 - n^{1/q} \left( \sum_{i=1}^n \frac{|a_{n-i}|^p}{|z|^{ip}} \right)^{1/p} \right\}.$$

If  $L_p \geq 1$ ,  $\max(L_p, L_p^{1/n}) = L_p$ . Let  $|z| \geq 1$ . Then  $1/|z|^{ip} \leq 1/|z|^p$  ( $i = 1, 2, \dots, n$ ). Hence (1) implies that if  $|z| > L_p$  then

$$|p(z)| \geq |z|^n \left\{ 1 - \frac{n^{1/q}}{|z|} \left( \sum_{i=0}^{n-1} |a_i|^p \right)^{1/p} \right\} = |z|^n \left( 1 - \frac{L_p}{|z|} \right) > 0.$$

Again if  $L_p \leq 1$ ,  $\max(L_p, L_p^{1/n}) = L_p^{1/n}$ . Let  $|z| \leq 1$ . Then  $1/|z|^{ip} \leq 1/|z|^{np}$  ( $i = 1, 2, \dots, n$ ). Hence, by (1), if  $|z| > L_p^{1/n}$  then

$$\begin{aligned} |p(z)| &\geq |z|^n \left\{ 1 - \frac{n^{1/q}}{|z|^n} \left( \sum_{i=1}^n |a_{n-i}|^p \right)^{1/p} \right\} \\ &= |z|^n \left\{ 1 - \frac{n^{1/q}}{|z|^n} \left( \sum_{i=0}^{n-1} |a_i|^p \right)^{1/p} \right\} \\ &= |z|^n \left( 1 - \frac{L_p}{|z|^n} \right) > 0. \end{aligned}$$

Hence  $p(z)$  does not vanish for  $|z| > \max(L_p, L_p^{1/n})$  and the theorem follows. The limit in Theorem 1 is attained by

$$p(z) = z^n - \frac{1}{n} (z^{n-1} + z^{n-2} + \dots + z + 1)$$

since

$$L_p = n^{1/q} \left( \sum_{i=0}^{n-1} \frac{1}{n^p} \right)^{1/p} = n^{1/q} \left( \frac{n}{n^p} \right)^{1/p} = \frac{n^{1/p} \cdot n^{1/q}}{n} = 1$$

and 1 is a zero of  $p(z)$ .

Letting  $q \rightarrow \infty$ , it follows that all the zeros of  $p(z)$  lie in

$$(2) \quad |z| \leq \max(L_1, L_1^{1/n}) \quad \text{where } L_1 = \sum_{i=0}^{n-1} |a_i|.$$

Applying this result to  $(1-z)p(z)$  we obtain the theorem of Montel and Marty mentioned above.

**THEOREM 2.** *If  $0 < a_{i-1} \leq ka_i$ ,  $k > 0$ , then all the zeros of  $P(z) = a_0 + a_1z + \dots + a_nz^n$  lie in  $|z| \leq \max(M, M^{1/n})$  where*

$$M = \frac{(a_0 + a_1 + \dots + a_{n-1})}{a_n} (k-1) + k.$$

*Proof.* Consider

$$\begin{aligned} F(z) &= (k - z)P(z) = (k - z)(a_0 + a_1z + \cdots + a_nz^n) \\ &= ka_0 + (ka_1 - a_0)z + (ka_2 - a_1)z^2 + \cdots + (ka_n - a_{n-1})z^n - a_nz^{n+1}. \end{aligned}$$

Applying (2) to the polynomial  $F(z)/a_n$  we find that

$$\begin{aligned} L_1 &= \frac{\sum_{i=0}^n |ka_i - a_{i-1}|}{a_n} = \frac{k(a_0 + a_1 + \cdots + a_n) - (a_0 + a_1 + \cdots + a_{n-1})}{a_n} \\ &= \frac{(k-1)(a_0 + a_1 + \cdots + a_{n-1})}{a_n} + k = M \end{aligned}$$

and the theorem follows. Putting  $k=1$  in Theorem 2 we get the following result due to Kakeya [1, p. 106].

**THEOREM B.** *If  $0 < a_0 \leq a_1 \leq \cdots \leq a_n$ , then all the zeros of the polynomial  $a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$  lie in  $|z| \leq 1$ .*

This follows from the fact that  $M=1$ , hence  $\max(M, M^{1/n})=1$ .

#### Reference

1. M. Marden, The geometry of zeros, Amer. Math. Soc. Math. Surveys, No. 3, New York, 1949.

#### A NOTE CONCERNING FERMAT'S CONJECTURE

W. E. CHRISTILLES, St. Mary's University, San Antonio, Texas

This paper introduces some elementary results related to the famous unsolved conjecture of Fermat, that there exists no nontrivial solution in integers of the equation

$$(1) \quad x^n + y^n + z^n = 0$$

for  $n$  an odd integer  $> 2$ . It is sufficient to consider the equation

$$(2) \quad x^p + y^p + z^p = 0,$$

for  $p$  an odd prime. Theorem 2 (below) is a new proof of Stone's Theorem 1 [1]. In addition an extension of Stone's Theorem will be stated and proven.

Assume that equation (2) has a solution  $x=a$ ,  $y=b$ ,  $z=c$ . The following restrictions result in no loss of generality.

$$(3.1) \quad abc \neq 0$$

$$(3.2) \quad |abc| \neq 1$$

$$(3.3) \quad |a| \text{ and } |b| \text{ are not both unity.}$$

$$(3.4) \quad (a, b) = 1, (a, c) = 1, \text{ and } (b, c) = 1.$$

$$(3.5) \quad c < 0 < a < b < |c|.$$