As before, since $F(0) \leqq K(0)$, there is a point $X \in [0, 1]$ such that $F(X) = K(X)$, and the proof is completed.

It might be noted that if $(p) + S(a, b) \subset E_2$, Theorem 1 is valid for the multiplier $1/\sqrt{3}$ (which is *less* than 1). In general, however, 1 is the smallest multiplier of pf for which Theorem 1 is valid. This can be seen by considering the Minkowski space $M_2^{(1)}$. The elements of this space are ordered pairs of real numbers with distance of $x = (x_1, x_2)$ and $y = (y_1, y_2)$ taken as

$$xy = |x_1 - y_1| + |x_2 - y_2| .$$

Let $S(a, b)$ be contained in the interval joining $(-1, 0)$ and $(1, 0)$ and let $p = (0, 1)$. It is easily seen that if p, x, y form an equilateral triple with $x, y \in S(a, b)$ then x and y *must* be the points $(-1, 0)$ and $(1, 0)$. Hence $S(a, b)$ is the interval joining these two points; the foot f of p on $S(a, b)$ is the origin, and $\min[af, bf] = pf$.

A slightly more complicated example can be given, also in the $M_2^{(1)}$, which shows that 2 is the smallest multiplier of pf for which Theorem 2 is valid.

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LOCATION OF THE ZEROS OF POLYNOMIALS

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The following theorem is due to Montel and Marty [**1**, p. 107].

Theorem A. *All the zeros of the polynomial*

$$p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$$

lie in $|z| \leqq \max(L, L^{1/n})$ *where* L *is the length of the polygonal line joining in succession the points* $0, a_0, a_1, \cdots, a_{n-1}, 1$; *i.e.*

$$L = |a_0| + |a_1 - a_0| + \cdots + |a_{n-1} - a_{n-2}| + |1 - a_{n-1}| .$$

We prove

Theorem 1. *All the zeros of the polynomial of Theorem A lie in* $|z| \leqq R = \max(L_p, L_p^{1/n})$ *where*

$$L_p = n^{1/q} \left(\sum_{i=0}^{n-1} |a_i|^p \right)^{1/p}, \quad p^{-1} + q^{-1} = 1.$$

The bound is sharp.

Proof. We have

$$(1)\quad |p(z)| \geqq |z|^n\left(1 - \sum_{i=1}^{n} \frac{|a_{n-i}|}{|z|^i}\right) \geqq |z|^n\left\{1 - n^{1/q}\left(\sum_{i=1}^{n} \frac{|a_{n-i}|^p}{|z|^{ip}}\right)^{1/p}\right\}.$$

If $L_p \geqq 1$, $\max(L_p, L_p^{1/n}) = L_p$. Let $|z| \geqq 1$. Then $1/|z|^{ip} \leqq 1/|z|^p$ $(i=1, 2, \cdots, n)$. Hence (1) implies that if $|z| > L_p$ then

$$|p(z)| \geqq |z|^n\left\{1 - \frac{n^{1/q}}{|z|}\left(\sum_{i=0}^{n-1} |a_i|^p\right)^{1/p}\right\} = |z|^n\left(1 - \frac{L_p}{|z|}\right) > 0.$$

Again if $L_p \leqq 1$, $\max(L_p, L_p^{1/n}) = L_p^{1/n}$. Let $|z| \leqq 1$. Then $1/|z|^{ip} \leqq 1/|z|^{np}$ $(i=1, 2, \cdots, n)$. Hence, by (1), if $|z| > L_p^{1/n}$ then

$$\begin{aligned} |p(z)| &\geqq |z|^n\left\{1 - \frac{n^{1/q}}{|z|^n}\left(\sum_{i=1}^{n} |a_{n-i}|^p\right)^{1/p}\right\} \\ &= |z|^n\left\{1 - \frac{n^{1/q}}{|z|^n}\left(\sum_{0}^{n-1} |a_i|^p\right)^{1/p}\right\} \\ &= |z|^n\left(1 - \frac{L_p}{|z|^n}\right) > 0. \end{aligned}$$

Hence $p(z)$ does not vanish for $|z| > \max(L_p, L_p^{1/n})$ and the theorem follows. The limit in Theorem 1 is attained by

$$p(z) = z^n - \frac{1}{n}(z^{n-1} + z^{n-2} + \cdots + z + 1)$$

since

$$L_p = n^{1/q}\left(\sum_{0}^{n-1} \frac{1}{n^p}\right)^{1/p} = n^{1/q}\left(\frac{n}{n^p}\right)^{1/p} = \frac{n^{1/p}\cdot n^{1/q}}{n} = 1$$

and 1 is a zero of $p(z)$.

Letting $q \to \infty$, it follows that all the zeros of $p(z)$ lie in

$$(2)\qquad |z| \leqq \max(L_1, L_1^{1/n}) \quad \text{where } L_1 = \sum_{i=0}^{n-1} |a_i|.$$

Applying this result to $(1-z)p(z)$ we obtain the theorem of Montel and Marty mentioned above.

THEOREM 2. *If* $0 < a_{i-1} \leqq ka_i$, $k > 0$, *then all the zeros of* $P(z) = a_0 + a_1z + \cdots + a_nz^n$ *lie in* $|z| \leqq \max(M, M^{1/n})$ *where*

$$M = \frac{(a_0 + a_1 + \cdots + a_{n-1})}{a_n}(k-1) + k.$$

Proof. Consider

$$F(z) = (k - z)P(z) = (k - z)(a_0 + a_1 z + \cdots + a_n z^n)$$
$$= ka_0 + (ka_1 - a_0)z + (ka_2 - a_1)z^2 + \cdots + (ka_n - a_{n-1})z^n - a_n z^{n+1}.$$

Applying (2) to the polynomial $F(z)/a_n$ we find that

$$L_1 = \frac{\sum_{i=0}^{n} |ka_i - a_{i-1}|}{a_n} = \frac{k(a_0 + a_1 + \cdots + a_n) - (a_0 + a_1 + \cdots + a_{n-1})}{a_n}$$
$$= \frac{(k-1)(a_0 + a_1 + \cdots + a_{n-1})}{a_n} + k = M$$

and the theorem follows. Putting $k=1$ in Theorem 2 we get the following result due to Kakeya [1, p. 106].

THEOREM B. *If* $0<a_0\leqq a_1\leqq \cdots \leqq a_n$, *then all the zeros of the polynomial* $a_0+a_1z+a_2z^2+ \cdots +a_nz^n$ *lie in* $|z| \leqq 1$.

This follows from the fact that $M=1$, hence $\max(M, M^{1/n})=1$.

Reference

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A NOTE CONCERNING FERMAT'S CONJECTURE

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This paper introduces some elementary results related to the famous unsolved conjecture of Fermat, that there exists no nontrivial solution in integers of the equation

(1) $$x^n + y^n + z^n = 0$$

for n an odd integer >2. It is sufficient to consider the equation

(2) $$x^p + y^p + z^p = 0,$$

for p an odd prime. Theorem 2 (below) is a new proof of Stone's Theorem 1 [1]. In addition an extension of Stone's Theorem will be stated and proven.

Assume that equation (2) has a solution $x=a$, $y=b$, $z=c$. The following restrictions result in no loss of generality.

(3.1) $abc \neq 0$

(3.2) $|abc| \neq 1$

(3.3) $|a|$ and $|b|$ are not both unity.

(3.4) $(a, b,) = 1$, $(a, c) = 1$, and $(b, c) = 1$.

(3.5) $c < 0 < a < b < |c|$.