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Source: *The American Mathematical Monthly*, Vol. 72, No. 6 (Jun. - Jul., 1965), pp. 631-633

Published by: Mathematical Association of America

Stable URL: <http://www.jstor.org/stable/2313853>

Accessed: 20/09/2008 15:46

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*Proof.* Define  $M = \{x: x \leq F(x) \text{ and } x_0 \leq x \leq x_1\}$ . Since  $x_0 \in M$ ,  $M$  is nonempty. By Zorn's lemma there exists a maximal chain  $L \subset M$ . It is clear that  $x_0 \in L$ . By Lemma 3  $\sup L$  exists; thus, define  $u = \sup L$ . Since  $x \leq u$  for all  $x \in L$ , we have  $x \leq F(x) \leq F(u)$  for all  $x \in L$ . Therefore,  $u \leq F(u)$ . Since  $u \leq x_1$  and  $L$  is a maximal chain in  $M$ , we must have  $u \in L$ . Since  $F$  is isotone and  $u \leq x_1$ , we have  $F(u) \leq F(x_1) \leq x_1$ , and since  $F(u) \leq F(F(u))$ , we have  $F(u) \in M$ . Hence, by the maximality of  $L$  and the fact that  $u \leq F(u)$ , we must have  $F(u) = u$ . Q.E.D.

We should note that fixed point theorems for isotone mappings have been obtained by Tarski; see [1, p. 54] or [3].

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#### ON THE ZEROS OF POLYNOMIALS

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The main purpose of this paper is to apply Schwarz's lemma to the study of the location of the zeros of a class of polynomials.

Throughout this paper  $P(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$ . Let

$$M = \max_{|z|=1} |a_1 z^{n-1} + \dots + a_n| = \max_{|z|=1} |a_n z^{n-1} + \dots + a_1|.$$

We prove

**THEOREM 1.** *All the zeros of  $P(z)$  lie in  $|z| \leq M/|a_0|$  if  $|a_0| \leq M$ .*

*Proof.* Consider  $p(z) = a_0 + a_1 z + \dots + a_n z^n$ , then

$$(1) \quad |p(z)| \geq |a_0| - |a_1 z + a_2 z^2 + \dots + a_n z^n|.$$

The definition of  $M$  and Schwarz's lemma imply that

$$|a_1 z + a_2 z^2 + \dots + a_n z^n| \leq M|z| \quad \text{for } |z| \leq 1.$$

Hence, (1) implies that if  $|z| \leq 1$  then

$$(2) \quad |p(z)| \geq |a_0| - M|z|.$$

Thus in  $|z| \leq 1$ ,  $|p(z)| > 0$  if  $|z| < |a_0|/M$  ( $\leq 1$  since  $|a_0| \leq M$  by hyp.); hence  $p(z)$  does not vanish for  $|z| < |a_0|/M$ . Consequently all the zeros of  $p(z)$  lie in  $|z| \geq |a_0|/M$ . As  $P(z) = z^n p(1/z)$ , we conclude that all the zeros of  $P(z)$  lie in  $|z| \leq M/|a_0|$ . This proves the theorem.

**REMARK 1.** If  $|a_0| \geq M$ , it follows easily from (2) that all the zeros of  $P(z)$  lie in  $|z| \leq 1$ .

**REMARK 2.** The limit in Theorem 1 is attained by  $P(z) = -nz^n + z^{n-1} + \dots + z + 1$ .

REMARK 3. The bound obtained in Theorem 1 is in general better than the traditional  $(|a_1| + |a_2| + \cdots + |a_n|)/|a_0|$ .

COROLLARY 1. If  $a_k \geq 0$ ,  $k=1, 2, \dots, n$  and  $|a_0| \leq a_1 + a_2 + \cdots + a_n$ , then all the zeros of  $P(z)$  lie in  $|z| \leq (a_1 + a_2 + \cdots + a_n)/|a_0|$ .

In particular all the zeros of  $R(z) = \pm S_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$  lie in  $|z| \leq 1$ , where  $S_0 = a_1 + a_2 + \cdots + a_n$ .

A well-known theorem of Enestrom andakeya ([1], p. 106) states that if  $a_0 \geq a_1 \geq a_2 \geq \cdots \geq a_{n-1} \geq a_n > 0$ , then all the zeros of  $P(z)$  lie in  $|z| \leq 1$ .

We show that it can be deduced easily from Corollary 1. Let

$$\begin{aligned} F(z) &= (1-z)P(z) = (1-z)(a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n) \\ &= -a_0 z^{n+1} + (a_0 - a_1)z^n + (a_1 - a_2)z^{n-2} + \cdots + (a_{n-1} - a_n)z + a_n \\ &= -a_0 z^{n+1} + Q(z). \end{aligned}$$

The hypothesis of the Enestrom-akeya theorem implies that  $Q(z)$  is a polynomial with nonnegative coefficients and the sum of the coefficients is clearly  $a_0$ . Hence by Corollary 1 all the zeros of  $F(z)$  lie in  $|z| \leq 1$ . As all the zeros of  $P(z)$  are also the zeros of  $F(z)$  we have proved the Enestrom-akeya Theorem.

THEOREM 2. Let  $r$  be the modulus of the zeros of largest modulus of  $P(z)$  and  $M' = \max_{|z|=r} |a_0 z^{n-1} + a_1 z^{n-2} + \cdots + a_{n-1}|$ . Then all the zeros of  $P(z)$  lie in the ring-shaped region  $r|a_n|/M' \leq |z| \leq r$  if  $|a_n| \leq M'$ .

Proof.  $P(z) = a_n + a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z$  implies, by Schwarz's lemma, that if  $|z| \leq r$  then

$$|P(z)| \geq |a_n| - |a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z| \geq |a_n| - \frac{M'|z|}{r}.$$

Hence  $|P(z)| > 0$  if  $|z| < r|a_n|/M'$  ( $\leq r$  since  $|a_n| \leq M'$  by hypothesis). Hence all the zeros of  $P(z)$  lie in the region  $r|a_n|/M' \leq |z| \leq r$ .

COROLLARY 2. If  $a_k \geq 0$ ,  $k=1, 2, 3, \dots, n-1$ ,  $a_0 > 0$ , and if  $|a_n| \leq a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r$ , then all the zeros of  $P(z)$  lie in

$$\frac{r|a_n|}{a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r} \leq |z| \leq r.$$

With the help of Corollary 2 we can restate the Enestrom-akeya Theorem in the following form:

THEOREM 3. If  $a_0 \geq a_1 \geq \cdots \geq a_{n-1} \geq a_n > 0$ , then all the zeros of  $P(z)$  lie in the ring-shaped region  $a_n/(a_0 + a_1 + \cdots + a_{n-1}) \leq |z| \leq 1$ .

The lower limit is attained by  $P(z) = z + 1$ . The following theorem can be easily deduced from the Enestrom-akeya Theorem ([1], p. 106).

THEOREM a. All the zeros of  $P(z)$  having real positive coefficients lie in  $|z| \leq \rho$ , where

$$(3) \quad \rho = \max \left( \frac{a_1}{a_0}, \frac{a_2}{a_1}, \dots, \frac{a_{n-1}}{a_{n-2}}, \frac{a_n}{a_{n-1}} \right).$$

Clearly  $\rho^n \geq a_n/a_0$  or  $a_0\rho^n \geq a_n$ . Hence,

$$M' = \max_{|z|=\rho} |a_0z^n + a_1z^{n-1} + \dots + a_{n-1}z| = a_0\rho^n + a_1\rho^{n-1} + \dots + a_{n-1}\rho > a_n.$$

Hence, applying Corollary 2 to  $P(z)$ , we can restate this theorem in the following form:

THEOREM 4. If  $\rho$  is given by (3) then all the zeros of  $P(z)$  having real positive coefficients lie in the ring-shaped region

$$\frac{\rho a_n}{a_0\rho^n + a_1\rho^{n-1} + \dots + a_{n-1}\rho} \leq z \leq \rho.$$

THEOREM 5. If  $a_0 \geq a_1 \geq \dots \geq a_{n-1} \geq a_n > 0$ , then the number of zeros of  $P(z)$  in  $|z| \leq \frac{1}{2}$  does not exceed

$$1 + \frac{1}{\log 2} \log \frac{a_0}{a_n}.$$

*Proof.* Consider

$$\begin{aligned} F(z) &= (1-z)(a_0z^n + a_1z^{n-1} + \dots + a_{n-1}z + a_n) \\ &= -a_0z^{n+1} + (a_0 - a_1)z^n + (a_1 - a_2)z^{n-1} + \dots + (a_{n-1} - a_n)z + a_n. \end{aligned}$$

For  $|z| \leq 1$ ,

$$\left| \frac{F(z)}{F(0)} \right| \leq \frac{a_0 + (a_0 - a_1) + \dots + (a_{n-1} - a_n) + a_n}{a_n} = \frac{2a_0}{a_n}.$$

If  $f(z)$  is regular,  $f(0) \neq 0$ , and  $|f(z)| \leq M$  in  $|z| \leq 1$ , then ([2], p. 171) the number of zeros of  $f(z)$  in  $|z| \leq 1/2$  does not exceed  $\{\log M/|f(0)|\}/\log 2$ . Therefore, if  $n(1/2)$  denotes the number of zeros of  $F(z)$  in  $|z| \leq 1/2$  then from above

$$n(\tfrac{1}{2}) \leq \log \frac{2a_0}{a_n} / \log 2 = 1 + \frac{1}{\log 2} \log \frac{a_0}{a_n}.$$

As the number of zeros of  $P(z)$  in  $z \leq 1/2$  is also equal to  $n(\frac{1}{2})$  the theorem is proved.

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