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1. INTRODUCTION. Descartes proposed the following rule in *La Geometrie* (1637):

> We can determine also the number of true and false roots that an equation can have, as follows: An equation can have as many true roots as it contains changes of sign, from + to − or from − to +; and as many false roots as the number of times two + signs or two − signs are found in succession [**3**, p. 160].

Descartes did not provide a proof for this rule, which forms the basis of the proposition now known as Descartes's Rule of Signs [**1**],[**6**]. This rule yields an upper bound for the number of positive (true) roots of a given polynomial and an upper bound for the number of negative (false) roots by counting variations and permanences in the sequence of plus and minus signs obtained from the coefficients of the polynomial when written in descending order: $p(x) = \sum_{i=0}^{n} a_{n-i}x^{n-i}$. For example, the polynomial $p(x) = x^4 - x^3 - 19x^2 + 49x - 30$ has the sequence of signs $+ - - + -$. This indicates three positive roots by the three variations (or changes) in sign, and one negative root in view of the one permanence, when the sign does not change. Descartes himself knew that his rule established only upper bounds on the number of positive and negative real roots, and Newton reiterated this point, stating that in many cases some of the roots will be "impossible." To Newton, "impossible" meant complex but not real, what the term "imaginary" signifies in this context today.

Newton devised a rule that provides a lower bound for the number of imaginary roots of a polynomial by considering variations and permanences in a sequence of signs obtained from the polynomial. In spirit then, this is similar to Descartes's Rule. However, Newton obtained his sequence of signs from the polynomial in quite a different, and more involved, manner. Newton discovered the rule in approximately 1666 and later inserted a brief account of it into his Lucasian lectures of October 1681, which are printed in the *Arithmetica Universalis* (1707).

2. NEWTON'S INCOMPLETE AND COMPLETE RULES. Consider a polynomial

$$p(x) = \binom{n}{0}a_n x^n + \binom{n}{1}a_{n-1}x^{n-1} + \binom{n}{2}a_{n-2}x^{n-2} + \cdots + \binom{n}{n-1}a_1 x + \binom{n}{n}a_0$$

with real coefficients that is written in the indicated form. The *simple elements* of $p(x)$ are defined as $a_n, a_{n-1}, a_{n-2}, \dots, a_0$, while the *quadratic elements* of $p(x)$ are defined to be $Q_n, Q_{n-1}, Q_{n-2}, \dots, Q_0$, where $Q_n = a_n^2$, $Q_{n-1} = a_{n-1}^2 - a_n a_{n-2}$, $Q_{n-2} = a_{n-2}^2 - a_{n-1}a_{n-3}, \dots, Q_1 = a_1^2 - a_2 a_0$, $Q_0 = a_0^2$.

Theorem 2.1 (Newton's Incomplete Rule). *Suppose that the quadratic elements for a polynomial $p(x)$ are all nonzero. Then the number of variations in sign in the sequence $Q_n, Q_{n-1}, Q_{n-2}, \dots, Q_0$ furnishes a lower bound for the number of imaginary roots of $p(x)$.*

Note that this bound is necessarily an even number, for both Q_n and Q_0 are positive. This makes sense, since complex roots come in conjugate pairs, as Newton knew. In what follows, we focus on the generic situation of nonzero simple elements and nonzero quadratic elements. However, we discuss the possibility that $a_i = 0$ or $Q_i = 0$ in section 6.

As an example, consider $p(x) = x^5 - 5x^4 + 4x^3 - 2x^2 - 5x - 4$. The simple elements are computed to be $1, -5/5, 4/10, -2/10, -5/5, -4$, which produce the following quadratic elements: $1, 6/10, -4/100, 44/100, 2/10, 16$. The resulting sequence of signs is $+ + - + ++$, which indicates at least two imaginary roots by Newton's Incomplete Rule. The following corollaries demonstrate that in some cases the rule provides knowledge of the existence of a single pair of imaginary roots without the labor of creating the sequence of signs.

Corollary 2.2. *For a polynomial $p(x) = \sum_{i=0}^{n} p_{n-i}x^{n-i}$, the relation*

$$p_r^2 < p_{r+1} \cdot p_{r-1}$$

for some r indicates the existence of a pair of imaginary roots.

Proof. Recall that

$$\begin{aligned} Q_r &= \frac{p_r^2}{\binom{n}{r}^2} - \frac{p_{r+1}}{\binom{n}{r+1}} \cdot \frac{p_{r-1}}{\binom{n}{r-1}} \\ &= \frac{1}{\binom{n}{r}^2} \left[p_r^2 - \left(\frac{r+1}{n-r}\right)\left(\frac{n-r+1}{r}\right) p_{r+1} \cdot p_{r-1} \right]. \end{aligned}$$

Up to sign we can ignore the first constant, and using

$$\left(\frac{r+1}{r}\right)\left(\frac{n-r+1}{n-r}\right) > 1,$$

we have

$$p_r^2 - \left(\frac{r+1}{n-r}\right)\left(\frac{n-r+1}{r}\right) p_{r+1} \cdot p_{r-1} < p_r^2 - p_{r+1} \cdot p_{r-1},$$

which is negative by hypothesis. Thus Q_r is negative. Again, since both Q_n and Q_0 are positive, the sequence of quadratic elements will have at least two variations in sign. Newton's Incomplete Rule guarantees at least one pair of imaginary roots. ∎

For example, the polynomial $p(x) = x^3 - 2x^2 + 4x - 16$ has a pair of imaginary roots because $4^2 < -2 \cdot -16$.

Corollary 2.3. *For a polynomial $p(x) = \sum_{i=0}^{n} p_{n-i}x^{n-i}$, the fact that both $|p_r| < |p_{r+1}|$ and $|p_r| < |p_{r-1}|$ hold for some r, with p_{r+1} and p_{r-1} of the same sign, indicates the existence of a pair of imaginary roots.*

Proof. The coefficients satisfy the condition of Corollary 2.2. ∎

As an example, the polynomial $p(x) = x^3 - 3x^2 + 2x - 4$ has a pair of imaginary roots because $|2| < |-3|$ and $|2| < |-4|$.

Because Newton's Incomplete Rule furnishes a lower bound for the number of imaginary roots possessed by a polynomial, an upper bound for the number of real roots is also obtained. In fact, Newton improved upper bounds derived by application of the Descartes Rule. We can ascertain that a pair of imaginary roots indicated by certain quadratic elements is hidden among the number of positive real roots or negative real roots predicted by Descartes's Rule by considering variations and permanences of the corresponding simple elements. This is Newton's Complete Rule, in which both the simple and quadratic sequences are examined. We write the simple elements as a horizontal sequence on top, and the quadratic elements as a horizontal sequence on the bottom:

$$\begin{array}{cccc} a_n & a_{n-1} & \cdots & a_0 \\ Q_n & Q_{n-1} & \cdots & Q_0. \end{array} \tag{1}$$

We concentrate on the associated pairs of consecutive elements,

$$\begin{array}{cc} a_{r+1} & a_r \\ Q_{r+1} & Q_r. \end{array}$$

We want to consider the possible sign change in the top pair and the possible sign change in the bottom pair, giving rise to four possibilities. We use the notation vV, vP, pV, and pP, where the first character always refers to the behavior of signs on the top and the second character refers to the behavior of signs on the bottom. The letters v and V denote a variation in sign; the letters p and P denote a permanence in sign. Thus, vV denotes a change in sign from the top pair a_{r+1} a_r and a change in sign from the associated bottom pair Q_{r+1} Q_r. The symbol vP denotes a change in sign from the top pair a_{r+1} a_r but a permanence in sign from the bottom pair Q_{r+1} Q_r. The symbols pV and pP have analogous meanings.

Theorem 2.4 (Newton's Complete Rule). *Suppose that the simple and quadratic elements for a polynomial $p(x)$ are all nonzero and are displayed as in (1). Then the total number of double permanences, written $\sum pP$, is an upper bound for the number of negative roots of $p(x)$, and the total number of variation-permanences, written $\sum vP$, is an upper bound for the number of positive roots.*

The second conclusion of the theorem follows from the first by considering $p(-x)$. As a corollary, the total number of real roots is less than or equal to the sum of double permanences and variation-permanences, i.e., to the total number $\sum P$ of permanences in the sequence of quadratic elements. Hence, the total number of imaginary roots is greater than or equal to $n - \sum P = \sum V$, the total number of variations in the sequence of quadratic elements. This is of course Newton's Incomplete Rule, which is now seen to be subsumed under the Complete Rule.

In our earlier example, $p(x) = x^5 - 5x^4 + 4x^3 - 2x^2 - 5x - 4$, the associated sequences of simple and quadratic elements are $\begin{array}{cccccc} + & - & + & - & - & - \\ + & + & - & + & + & + \end{array}$, revealing one variation-permanence and two double permanences, and thus indicating one positive root and at most two negative roots. Descartes's Rule predicts at most three positive roots and at most two negative roots. This particular quintic actually has one positive root and four imaginary roots.

Newton's Complete Rule gives full knowledge of the nature of the roots for any polynomial that has no more imaginary roots than the rule discloses, unlike the case of the aforementioned quintic. Newton was of the opinion that this latter scenario rarely happens, but nonetheless provided the example of $p(x) = x^3 - 3a^2x - 3a^3$, which produces quadratic elements with sign pattern $++++$, thereby concealing the two imaginary roots found by employing Cardan's resolution of the cubic. Newton added that in such cases one could alter the roots by a change of variable from x to $x+p$ [**8**, p.529]:

> And if there be any impossible roots it will rarely happen that they shall not be discovered in two or three such trials. Nor can there be an equation whose impossible roots may not be thus discovered.

This cryptic comment forms the key to Sylvester's proof of Newton's Complete Rule wherein he meticulously charts how the simple and quadratic sign patterns change for the polynomials $p(x+\lambda)$ as λ moves continuously through $\mathbb{R}$. This technique is analogous to one devised by Fourier, who in 1796 proved a generalization of Descartes's Rule that was published posthumously in 1831 [**4**].

3. WITHOUT A PROOF. Newton offered no proof for his incomplete or complete rules. It is not difficult to show that a variation in the signs produced by Newton's Incomplete Rule indicates a pair of imaginary roots [**7**]. George Campbell published a proof in 1728 [**2**], and Colin Maclaurin treated the same matter as part of a lengthy letter to Martin Folkes dealing with roots of equations. The letter was later published in 1730 [**5**]. However, it does not follow that each occurrence of a double sign change indicates a separate pair of imaginary roots. Edward Waring, who in 1760 became Lucasian Professor of Mathematics at Cambridge, pointed this out in 1782, prior to which many believed Campbell's restricted result proved Newton's Incomplete Rule. Subsequently, many attempted rigorous justification of the incomplete rule, and yet a true proof was not forthcoming until J.J. Sylvester provided one in 1865. Sylvester entitled his paper "On an Elementary Proof and Generalization of Sir Isaac Newton's Hitherto Undemonstrated Rule for the Discovery of Imaginary Roots"[**7**]. Sylvester mentioned that all previous work amounted to showing only the existence of a single pair of complex roots once a variation was produced, with no progress at all concerning Newton's Complete Rule. He furthermore devoted a postscript to tearing apart a supposed proof of Newton's Incomplete Rule given twenty years earlier by a Professor J. R. Young, who claimed priority over Sylvester. Sylvester characterized the work as "a so-called proof," and "such stuff as dreams are made of."

We now describe the notation necessary for Sylvester's generalization of Newton's Complete Rule. Form the simple and quadratic elements for the new polynomial $p(x+\lambda)$ and write $\sum pP(\lambda)$ for the total number of double permanences therein, and similarly for $\sum vV(\lambda)$, $\sum pV(\lambda)$, and $\sum vP(\lambda)$.

Theorem 3.1 (Sylvester's generalization of Newton's Complete Rule). *Let μ and ν be any two real numbers with $\mu > \nu$. Then*

$$\sum pP(\mu) - \sum pP(\nu) = (\nu, \mu) + 2k,$$

where (ν, μ) denotes the total number of real roots of $p(x)$ between ν and μ, counted with multiplicities, and k is some nonnegative integer.

As a comment, recall that the graph of the polynomial $p(x+\mu)$ can be thought of as the graph of $p(x)$ shifted μ units to the left. Similarly for $p(x+\nu)$. Now as μ exceeds ν, $p(x+\mu)$ will have at least as many negative real roots as $p(x+\nu)$, perhaps more. Sylvester's theorem quantifies how the change in double permanences for the simple and quadratic sequences associated with these two polynomials is commensurate with the change in the nature of roots. Fourier's earlier result states that

$$\sum p(\mu) - \sum p(\nu) = (\nu, \mu) + 2k.$$

Thus, Fourier only considered the sign pattern of the simple elements, i.e., the pattern examined in Descartes's Rule.

To illustrate Sylvester's theorem, consider the polynomial $p(x) = x^3 + 2x^2 + 4x + 1$. We compute the associated sequence of signs formed from the simple elements and quadratic elements for the polynomials $p(x + \lambda)$, where $\lambda = -1, 0$, and 2.

$\lambda = -1, (\Sigma pP = 0)$	$\lambda = 0, (\Sigma pP = 1)$	$\lambda = 2, (\Sigma pP = 1)$
$+ \; - \; + \; -$	$+ \; + \; + \; +$	$+ \; + \; + \; +$
$+ \; - \; + \; +$	$+ \; - \; + \; +$	$+ \; - \; - \; +$

According to Sylvester's theorem, with $\mu = 0$ and $\nu = -1$, we should gain as many double permanences as roots for $p(x)$ in $(-1, 0)$, or more by an even number. The pattern at $\lambda = -1$ changes from no double permanences to one double permanence at $\lambda = 0$, indicating that $p(x)$ has only one root in $(-1, 0)$. (In fact, Newton's Complete Rule indicates that $p(x)$ has one negative root and two imaginary roots. The pattern for $p(x)$ corresponds to $\lambda = 0$.) There is also a change in pattern when considering $\mu = 2$ and $\nu = 0$, but the number of double permanences doesn't change.

As an additional example, let $p(x) = x^4 + 2x^3 + x^2 + 2x + 3$. The associated sequences of signs for $p(x + \lambda)$ for several choices of λ are shown.

$\lambda = -3$	$\lambda = -2$	$\lambda = -1$	$\lambda = 0$	$\lambda = 6$	$\lambda = 14$
$+ - + - +$	$+ - + - +$	$+ - + + +$	$+ + + + +$	$+ + + + +$	$+ + + + +$
$+ + + + +$	$+ + + - +$	$+ + + + +$	$+ + - - +$	$+ + + - +$	$+ + + + +$

The polynomial $p(x)$ has four imaginary roots, and thus, by Sylvester's theorem, we expect to see the number of double permanences increase by even numbers only. For example, with $\mu = 14$ and $\nu = -3$, we see an increase of four double permanences.

Corollary 3.2. *Newton's Complete Rule is valid.*

Proof. First, note that

$$\sum pP(0) - \sum pP(-\infty) = \sum pP(0),$$

where

$$\sum pP(-\infty) = \lim_{\lambda \to -\infty} \sum pP(\lambda).$$

This follows because expansion of $p(x + \lambda)$ produces

$$a_n x^n + n(a_n \lambda + a_{n-1})x^{n-1} + \frac{n(n-1)}{2}(a_n \lambda^2 + 2a_{n-1}\lambda + a_{n-2})x^{n-2} + \cdots.$$

For λ sufficiently negative, the rth coefficient will carry the same sign as $a_n \lambda^r$, thereby rendering $\sum pP(-\infty) = 0$, for the simple elements alternate in sign. (Alternatively, we could use Taylor's theorem to show that the coefficients of $p(x + \lambda)$ are just $p^{(r)}(\lambda)$.) Observe that $\sum pP(0)$ corresponds to our original $\sum pP$. By Sylvester's theorem we have

$$\sum pP(0) = \sum pP(0) - \sum pP(-\infty) \geq (-\infty, 0).$$

Finally, changing $p(x)$ to $p(-x)$ reverses the permanences and variations of the simple elements, thus yielding $\sum vP(0) \geq (0, \infty)$. ■

4. SOME PRELIMINARIES. The content of Sylvester's theorem lies in comparing the associated sequences of simple and quadratic elements of the polynomials $p(x + \lambda)$ as λ varies continuously over $\mathbb{R}$. Given a polynomial $p(x)$, we first wish to find convenient expressions for the simple and quadratic elements of $p(x + \lambda)$. Later in this section we use these expressions to classify the possible cases of simple or quadratic elements equaling zero.

Recall that by Taylor's theorem we can expand $p(x)$ about $x = \lambda$:

$$p(x) = p(\lambda) + p'(\lambda)(x - \lambda) + \frac{1}{2}p''(\lambda)(x - \lambda)^2 + \cdots + \frac{1}{n!}p^{(n)}(\lambda)(x - \lambda)^n.$$

Replacing x with $x + \lambda$ then leads to

$$\begin{aligned} p(x + \lambda) &= p(\lambda) + p'(\lambda)x + \frac{1}{2}p''(\lambda)x^2 + \cdots + \frac{1}{n!}p^{(n)}(\lambda)x^n = \frac{1}{n!}p^{(n)}(\lambda)x^n \\ &\quad + \frac{1}{(n-1)!}p^{(n-1)}(\lambda)x^{(n-1)} + \cdots + \frac{1}{2}p''(\lambda)x^2 + p'(\lambda)x + p(\lambda). \end{aligned}$$

The simple elements of $p(x + \lambda)$ are therefore

$$\begin{gathered} \frac{1}{n!}p^{(n)}(\lambda), \frac{1}{n}\frac{1}{(n-1)!}p^{(n-1)}(\lambda), \frac{1 \cdot 2}{n(n-1)}\frac{1}{(n-2)!}p^{(n-2)}(\lambda), \ldots, \\ \frac{1}{\binom{n}{k}}\frac{1}{(n-k)!}p^{(n-k)}(\lambda), \ldots, \frac{1}{n}p'(\lambda), p(\lambda). \end{gathered}$$

Multiplying each of these terms by the positive quantity $n!$ simplifies these elements without changing either the roots or the permanences and variations:

$$p^{(n)}(\lambda), p^{(n-1)}(\lambda), 2!p^{(n-2)}(\lambda), 3!p^{(n-3)}(\lambda), 4!p^{(n-4)}(\lambda), \ldots, n!p(\lambda). \tag{2}$$

Forming the quadratic elements from this and afterwards simplifying each successive term by dividing by $(1!)^2, (2!)^2, (3!)^2, (4!)^2, \ldots$, respectively (recall that we're interested in the signs of these terms only), we arrive at

$$Q_n(\lambda), Q_{n-1}(\lambda), Q_{n-2}(\lambda), Q_{n-3}(\lambda), Q_{n-4}(\lambda), \ldots, Q_0(\lambda),$$

where

$$Q_n(\lambda) = (p^{(n)}(\lambda))^2, Q_0(\lambda) = (p(\lambda))^2, \tag{3}$$

and

$$Q_k(\lambda) = (p^{(k)}(\lambda))^2 - \gamma_k p^{(k-1)}(\lambda)p^{(k+1)}(\lambda) \tag{4}$$

with $\gamma_k = (n - k + 1)/(n - k)$ when $1 \leq k \leq n - 1$.

Thus, the expressions (2), (3), and (4) for these simple and quadratic elements are polynomials (formed from the derivatives of our original $p(x)$) that are evaluated at λ. By continuity there will be no change in sign of any particular simple or quadratic element as λ changes, except perhaps when this element becomes zero. Suppose this happens at a particular λ. That is, suppose $p^{(k)}(\lambda)$ (a simple element of the polynomial $p(x+\lambda)$) or $Q_k(\lambda)$ (a quadratic element of $p(x+\lambda)$) is zero. We organize the occurrence of zeros into the following five cases by using (3) and (4), which describe how the quadratic elements are formed from the simple elements (note that $p^{(k+1)}$ denotes $p^{(k+1)}(\lambda)$ in what follows):

a. An intermediate simple element becomes zero, without its two adjacent simple elements becoming zero. The associated three quadratic elements are necessarily nonzero by (4). As part of the associated sequences of simple and quadratic elements at λ (i.e., for the polynomial $p(x+\lambda)$) we have the following pattern:

$$\begin{matrix} p^{(k+1)} & 0 & p^{(k-1)} \\ Q_{k+1} & Q_k & Q_{k-1}. \end{matrix} \tag{5}$$

b. An intermediate quadratic element becomes zero, without its two adjacent quadratic elements becoming zero. The three associated simple elements are nonzero by (4), and we see the following as part of the associated sequences:

$$\begin{matrix} p^{(k+1)} & p^{(k)} & p^{(k-1)} \\ Q_{k+1} & 0 & Q_{k-1}. \end{matrix} \tag{6}$$

c. The last simple element becomes zero, perhaps preceded by several consecutive zero simple elements. This is the case when λ is a root of any multiplicity i less than n. Using (3) and (4) we see that the associated sequences of simple and quadratic elements end in the following manner:

$$\begin{matrix} p^{(i)} & 0 & \cdots & 0 & 0 \\ Q_i & 0 & \cdots & 0 & 0. \end{matrix} \tag{7}$$

d. Several consecutive intermediate simple elements become zero at λ, forcing the associated quadratic elements to become zero as well. We observe the following pattern as part of our associated sequences:

$$\begin{matrix} p^{(k+1)} & 0 & 0 & \cdots & p^{(k-i)} \\ Q_{k+1} & 0 & 0 & \cdots & Q_{k-i}. \end{matrix} \tag{8}$$

e. Several consecutive intermediate quadratic elements become zero at λ, with all the associated simple elements still nonzero. Under this assumption, if one of these simple elements were to become zero at λ, then the remaining simple elements would necessarily become zero as well. We described this situation in (d). As part of the sequences we find:

$$\begin{matrix} p^{(k+1)} & p^{(k)} & p^{(k-1)} & \cdots & p^{(k-i)} \\ Q_{k+1} & 0 & 0 & \cdots & Q_{k-i}. \end{matrix} \tag{9}$$

No extra consideration is needed when several of these cases occur at the same λ, for in all five of the cases Sylvester considered a zero element or a group of consecutive zero elements with their associated elements and nonzero adjacent elements, as shown in (5), (6), (7), (8), and (9). Thus, we do not overcount the double permanences or variation permanences in one case because they do not appear in another concurrent case.

Also, in the proof for each of these cases, Sylvester considered the general situation where zeros of the polynomial expressions that yield the simple and quadratic elements upon evaluation are necessarily isolated. In section 6 of this article we discuss Sylvester's conclusion when some quadratic expressions vanish identically.

5. SYLVESTER'S INGENIUS PROOF. Sylvester claimed (as in Theorem 3.1) that the associated sequences of signs for the polynomials $p(x+\mu)$ and $p(x+\nu)$ differed in a precise way (that enabled Sylvester to validate Newton's earlier work). In fact, the kth simple (quadratic) element for the polynomial $p(x+\mu)$ will have the same sign as the corresponding kth element for $p(x+\nu)$ unless perhaps this element becomes zero for the polynomial $p(x+\lambda)$, where $\mu > \lambda > \nu$. Without loss of generality, let $\mu = \lambda + \epsilon$ and $\nu = \lambda - \epsilon$, where ϵ is small enough so that the polynomial expression (2) for the kth simple (quadratic (3)) element is zero in the interval $(\lambda - \epsilon, \lambda + \epsilon)$ only at λ.

In this section we present identities Sylvester used to show how the change from 0 (at λ) to + or $-$ (at $\lambda \pm \epsilon$) is completely determined by the signs of the nonzero adjacent elements at λ and the sign of ϵ. Amazingly, no other details from the polynomial $p(x)$ are necessary. Then, we demonstrate how Sylvester used these identities in all of the cases (a) through (e) presented in section 4 to verify that the sequences of signs do indeed differ as prescribed in Theorem 3.1.

Sylvester deduced the following identities from the presented expressions (2), (3), and (4) for the simple and quadratic elements, Taylor's theorem, and the continuity of polynomials. It is upon these relations that the analysis rests. In the derivations for identities three and four that follow, Sylvester used the fact that $2 - \gamma_k = 1/\gamma_{k+1}$. Also, note that the approximations have error terms that tend to zero by Taylor's Remainder Theorem for polynomials (the remainder terms are $o(\epsilon)$ in identities (i) and (iii), $o(\epsilon^i)$ in identities (ii) and (iv)). Thus, the signs of the terms on either side of the approximation symbol are necessarily the same when ϵ is sufficiently small. Here are the approximations that Sylvester invoked:

i. $p^{(k)}(\lambda) = 0 \Rightarrow p^{(k)}(\lambda + \epsilon) \approx \epsilon p^{(k+1)}(\lambda)$

ii. $p^{(k)}(\lambda) = 0 = p^{(k+1)}(\lambda) = \cdots = p^{(k+i-1)}(\lambda) = 0$
$$\Rightarrow p^{(k)}(\lambda + \epsilon) \approx \frac{\epsilon^i}{i!} p^{(k+i)}(\lambda)$$

iii. $Q_k(\lambda) = 0 \Rightarrow Q_k(\lambda + \epsilon) \approx \epsilon \cdot \frac{d}{dx} Q_k(x)\Big|_{x=\lambda} = \dfrac{\epsilon}{\gamma_{k+1}} \dfrac{p^{(k)}(\lambda)}{p^{(k+1)}(\lambda)} Q_{k+1}(\lambda)$

iv. $Q_k(\lambda) = 0 = Q_{k+1}(\lambda) = \cdots = Q_{k+i-1}(\lambda)$
$$\Rightarrow Q_k(\lambda + \epsilon) \approx \frac{\epsilon^i}{i!\gamma_{k+1} \cdots \gamma_{k+i}} \frac{p^k(\lambda)}{p^{k+i}(\lambda)} Q_{k+i}(\lambda)$$

Case a. Suppose we witness the single vanishing of a simple element $p^{(k)}(\lambda)$ with $k \neq 0$ or n. The quadratic elements in (10) are determined by the upper elements and thus we have only $p^{(k+1)}$ and $p^{(k-1)}$ to assign freely a plus or minus sign. However, reversing all signs on top does not interchange a permanence and variation. Thus we actually have $2^2/2 = 2$ possibilities for what happens at λ. The boxed terms will be used to compute the signs before and after λ.

$$\begin{matrix} p^{(k+1)} & p^{(k)} & p^{(k-1)} \\ Q_{k+1} & Q_k & Q_{k-1} \end{matrix} = \begin{matrix} \boxed{+} & 0 & + \\ + & - & + \end{matrix} \quad \text{or} \quad \begin{matrix} \boxed{+} & 0 & - \\ + & + & +. \end{matrix} \tag{10}$$

Observe that all signs in (10) must remain constant through the transition, except possibly at the position occupied by the kth simple element. The corresponding signs for this position are deduced using identity (i), which says the sign of $p^{(k)}(\lambda + \epsilon)$ is given by multiplying the sign of ϵ and the boxed term.

Before this occurrence of zero (i.e., at $\lambda - \epsilon$) we calculate the corresponding elements to be:

$$\begin{matrix} p^{(k+1)} & p^{(k)} & p^{(k-1)} \\ Q_{k+1} & Q_k & Q_{k-1} \end{matrix} = \begin{matrix} + & - & + \\ + & - & + \end{matrix} \quad \text{or} \quad \begin{matrix} + & - & - \\ + & + & +. \end{matrix}$$

After the occurrence of zero (i.e., at $\lambda + \epsilon$) we calculate the corresponding elements to be:

$$\begin{matrix} p^{(k+1)} & p^{(k)} & p^{(k-1)} \\ Q_{k+1} & Q_k & Q_{k-1} \end{matrix} = \begin{matrix} + & + & + \\ + & - & + \end{matrix} \quad \text{or} \quad \begin{matrix} + & + & - \\ + & + & +. \end{matrix}$$

Thus, we see no net loss or gain of double permanences in proceeding from $\lambda - \epsilon$ to $\lambda + \epsilon$ in $\mathbb{R}$.

Case b. Suppose we witness the single vanishing of a quadratic element $Q_k(\lambda)$, where k differs from 0 and n. This zero element dictates by the definition of Q_k that the outer simple elements necessarily be of the same sign and that we have four positions to assign freely either a $+$ or $-$. However, as before we actually have only $2^4/4 = 4$ possibilities at λ (in this case the lower signs aren't determined by the triple above them):

$$\begin{matrix} \boxed{+} & \boxed{+} & + \\ \boxed{+} & 0 & - \end{matrix} \qquad \begin{matrix} \boxed{+} & \boxed{+} & + \\ \boxed{+} & 0 & + \end{matrix} \qquad \begin{matrix} \boxed{+} & \boxed{-} & + \\ \boxed{+} & 0 & - \end{matrix} \qquad \begin{matrix} \boxed{+} & \boxed{-} & + \\ \boxed{+} & 0 & +. \end{matrix}$$

The new sign corresponding to the kth quadratic position is deduced from identity (iii), which says this sign is obtained from the product of the boxed terms with the sign of ϵ.

Before this occurrence of zero (i.e., at $\lambda - \epsilon$) we calculate the corresponding elements to be:

$$\begin{matrix} + & + & + \\ + & - & - \end{matrix} \qquad \begin{matrix} + & + & + \\ + & - & + \end{matrix} \qquad \begin{matrix} + & - & + \\ + & + & - \end{matrix} \qquad \begin{matrix} + & - & + \\ + & + & +. \end{matrix}$$

After the occurrence of zero (i.e., at $\lambda + \epsilon$) we find that the corresponding elements are:

$$\begin{matrix} + & + & + \\ + & + & - \end{matrix} \qquad \begin{matrix} + & + & + \\ + & + & + \end{matrix} \qquad \begin{matrix} + & - & + \\ + & - & - \end{matrix} \qquad \begin{matrix} + & - & + \\ + & - & +. \end{matrix}$$

Thus, we see no net gain or loss for the first, third, and fourth forms, and a gain of two double permanences in the second form.

Case c. If λ is a root of multiplicity i for $p(x)$, then the final $i + 1$ elements of the associated sequences at λ,

$$\begin{array}{ccccc} p^{(i)} & p^{(i-1)} & \cdots & p' & p \\ Q_i & Q_{i-1} & \cdots & Q_1 & Q_0, \end{array}$$

become

$$\begin{array}{cccc} p^{(i)} & 0 & \cdots & 0 \;\; 0 \\ Q_i & 0 & \cdots & 0 \;\; 0, \end{array}$$

reducible to the single case

$$\begin{array}{ccccc} + & 0 & \cdots & 0 & 0 \\ + & 0 & \cdots & 0 & 0. \end{array}$$

At $\lambda - \epsilon$ we see

$$\begin{array}{ccccccc} + & - & + & - & \cdots & \pm & \mp \\ + & + & + & + & \cdots & + & +, \end{array}$$

and at $\lambda + \epsilon$ the form

$$\begin{array}{ccccccc} + & + & + & + & \cdots & + & + \\ + & + & + & + & \cdots & + & +, \end{array}$$

picking up i double permanences as we pass through a root of multiplicity i. The signs for the simple elements were computed using identity (ii), wherein the expression $\epsilon^{i-k} p^{(i)}(\lambda)/(i-k)!$ dictates that the sign comes from ϵ^{i-k}, which produces simple variations before transit (ϵ is negative) and simple permanences after transit (ϵ is positive). Already then we see that no double permanences occur at $\lambda - \epsilon$. As for the quadratic elements, note that $Q_i(x)$ is positive throughout the transition by continuity, and $Q_0(x)$ is positive at $\lambda - \epsilon$ and $\lambda + \epsilon$ because it is determined by the simple element above it by (3). A straightforward analysis using (4) shows that the intermediate quadratic elements remain positive throughout transit.

In summary, as we increase the parameter λ, the number of double permanences in the associated sequences of simple and quadratic elements increases by the number of real roots passed over, each counted with multiplicity, and any increase beyond that is given by an even integer. The remaining cases, cases (d) and (e), feature multiple terms vanishing concurrently. Sylvester mentioned that a small perturbation of the coefficients would change these singular cases to the generic one, while leaving the nature of the roots unaltered. However, a possible exception involves the passing of real roots to an imaginary pair, and rather than deal with this subtlety, he handled the singular cases as we have indicated. The conclusion still holds.

6. COMMENTS ON ZERO TERMS, SIMPLE AND QUADRATIC. In Sylvester's paper, there is no assignment of plus or minus signs when simple or quadratic elements for $p(x+\lambda)$ become zero. In fact the zero elements, all isolated occurences, mark potential transitions in sign, and counting double permanences at these transition points is unimportant. However, the initial polynomial $p(x)$, corresponding to $\lambda = 0$, should have sequences of signs if we wish to apply Newton's Rules. As is, Sylvester's theorem and Newton's Rules apply to polynomials $p(x)$ such that the associated simple and quadratic expressions, themselves polynomials, contain only isolated zeros and such that the initial simple and quadratic elements for $p(x)$ ($\lambda = 0$) are all nonzero.

However, Newton originally provided some ad hoc rules for the assignment of plus and minus signs to zero quadratic elements. Sylvester did not address these rules, but we now observe that the full strength of Sylvester's conclusion holds when we apply these rules to polynomials that contain identically zero quadratic expressions (simple expressions are never identically zero) or polynomials with zeros among the initial simple or quadratic elements. For brevity's sake we omit the proofs of these rules, but we remark that the arguments closely parallel those of Sylvester that we have presented earlier.

Rule A: Signs for the zeros of cases (a) and (b) (Newton). Assign any lone intermediate zero quadratic element the sign of $-$ and any lone intermediate zero simple element arbitrary sign. Arguing as in the proof of Sylvester's theorem for cases (a) and (b), we learn that

$$\sum pP(\lambda + \epsilon) - \sum pP(\lambda) = (\lambda + \epsilon, \lambda) + 2k$$

holds; a similar fact emerges when comparing $\sum pP(\lambda)$ and $\sum pP(\lambda - \epsilon)$. Moreover, we can also show that this assignment of signs forces the counts $\sum vP$ and $\sum pP$ to swap under the transformation $x \mapsto -x$, as required in the proof of Newton's Complete Rule (Corollary 3.2).

Rule B: Signs for the zeros of cases (d) and (e) (Newton). If consecutive intermediate zeros occur in the quadratic sequence due to simple zeros overhead (e.g., there are missing terms in the polynomial, i.e., we are in case (d) with $\lambda = 0$), assign consecutive quadratic zero elements alternately $-$ and $+$, beginning with the former, and in addition assign a $+$ to the final zero element if the simple elements adjacent to the group of zeros are of opposite signs. Once this is done, we assign an alternating pattern to the zero simple elements, beginning with sign opposite that of the preceeding nonzero element. Once again, we can show that Sylvester's conclusion holds and that the counts $\sum vP$ and $\sum pP$ are interchanged under the involution $x \mapsto -x$. We follow the same rule in the case of consecutive intermediate zeros in the quadratic sequence without simple zeros overhead (i.e., case (e)), but note that in this case simple elements must either agree in sign or form an alternating pattern, meaning that the proviso of reassigning a $+$ to the final zero element slated for a $-$ will never apply.

Rule C: Signs for identically zero quadratic elements. Suppose $p(x)$ possesses identically zero quadratic expressions Q_i ($Q_i(\lambda) = 0$ for all λ), with Q_k the last such term. We can show that this implies the existence of a sequence of consecutive identically zero quadratic expressions, beginning with Q_{n-1} and ending with Q_k, and moreover that

$$p(x) = (x - \alpha)^n + q(x) \tag{11}$$

for some polynomial $q(x)$ of degree $k - 2$. We assign the consecutive vanishing elements through Q_k the alternating pattern $- + - + - + \cdots$, once again ending with $+$ if the simple elements bordering the zero elements are of opposite sign. We examine the associated elements

$$\begin{array}{cccc} p^{(n)} & p^{(n-1)} & \cdots & p^{(k-2)} \\ Q_n & Q_{n-1} & \cdots & Q_{k-2} \end{array}$$

and study in particular the transition through $\lambda = \alpha$, i.e., the only value for which both $p^{(k-1)} = 0$ and $Q_{k-1} = 0$, as seen by computing $p^{(k-1)}$ and Q_{k-1} directly from (11).

As in Rule B, we assign the consecutive quadratic zeros an alternating pattern, ending with $+$ if the simple elements bordering the zero elements are of opposite sign. The consecutive simple zeros at $x = \alpha$ are also assigned an alternating pattern, beginning with the sign opposite that of the preceeding element (the leading coefficient of $p(x)$). Once again, we can show Sylvester's conclusion holds when comparing $\sum pP(0)$ and $\sum pP(\alpha)$, and when comparing $\sum pP(\alpha)$ and $\sum pP(\alpha+\epsilon)$. (If α were negative, we would consider $\sum pP(\alpha-\epsilon)$ instead of $\sum pP(\alpha+\epsilon)$.) Also, the counts $\sum vP$ and $\sum pP$ swap under the involution $x \mapsto -x$.

We note that in applying Newton's Rules the reader will not know whether a string of consecutive zero quadratic elements beginning with Q_{n-1} is the result of identically zero quadratic expressions, thus requiring Rule C, or merely isolated zeros of quadratic expressions that are not identically zero, when Rule B applies. But, because Rules B and C assign $+$ and $-$ signs in an identical manner, the distinction is unnecessary in practice. However, separate proofs are required for these two different situations.

In conclusion, Sylvester's theorem—and therefore Newton's Incomplete and Complete Rules—holds for all polynomials $p(x)$ such that $p(x)$ has a nonzero constant term and $p(x) \neq (x-\alpha)^n$. There is not much loss in generality in these two exceptions. In the former case, which corresponds to case (c) with $\lambda = 0$, we can rewrite $p(x) = x^i \widehat{p}(x)$ and apply Sylvester's theorem to $\widehat{p}(x)$. In the latter instance, $p(x) = (x-\alpha)^n$ is easily recognized in its expansion. In fact, this situation arises if and only if all intermediate quadratic expressions are identically zero. Complete knowledge of the nature of the roots of such a polynomial is immediate.

7. EXAMPLES OF SPECIAL CASES.

1. $p(x) = x^5 - 2x - 1$. In the associated sequences of simple and quadratic elements, the circled signs indicate zero elements that were assigned a plus or minus in accordance with the ad hoc rules given in section 6. Here we use Rule B to compute

$$\begin{array}{cccccc} + & \ominus & \oplus & \ominus & - & - \\ + & \ominus & \oplus & \oplus & + & +, \end{array}$$

indicating one positive root, two negative roots, and two imaginary roots—an accurate prediction for this polynomial.

2. $p(x) = (x-2)^5 - 2x + 3 = x^5 - 10x^4 + 40x^3 - 80x^2 + 78x - 29$. The associated sequences of simple and quadratic terms (Rule C) is

$$\begin{array}{cccccc} + & - & + & - & + & - \\ + & \ominus & \oplus & + & + & +, \end{array}$$

indicating three positive roots, no negative roots, and two imaginary roots, which again represents an accurate profile of the roots of this polynomial.

3. $p(x) = (x-2)^6 + 3x^2 - 2x + 1 = x^6 - 12x^5 + 60x^4 - 160x^3 + 243x^2 - 194x + 65$. The associated sequences of simple and quadratic terms (Rule C) is

$$\begin{array}{ccccccc} + & - & + & - & + & - & + \\ + & \ominus & \oplus & - & + & - & +, \end{array}$$

indicating six imaginary roots. Once more this accurately predicts the distribution of roots.

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