SOME REMARKABLE PROPERTIES OF SINC AND RELATED INTEGRALS

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Abstract. Using Fourier transform techniques, we establish inequalities for integrals of the form

$$\int_0^\infty \prod_{k=1}^n \frac{\sin(a_k x)}{x}\, dx.$$

We then give quite striking closed form evaluations of such integrals and finish by discussing various extensions and applications.

1. Introduction. Motivated by questions about the integral[1]

$$\mu := \int_0^\infty \prod_{k=1}^\infty \cos\left(\frac{x}{k}\right)\, dx, \tag{1}$$

we study the behaviour of integrals of the form

$$\int_0^\infty \prod_{k=1}^n \frac{\sin(a_k x)}{x}\, dx.$$

In Section 2 we use Fourier transform theory to establish monotonicity properties of these integrals as functions of the parameters. In Section 3, by direct methods, we give closed forms for these integrals and for similar integrals also incorporating cosine terms. In Section 4, we provide a very different proof of one of these results following an idea in an 1885 paper of Störmer [2]. Finally, in Section 5 we return to the study of (1).

2. Fourier cosine transforms and sinc integrals. Define

$$\mathrm{sinc}(x) := \frac{\sin x}{x},$$

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[1] Through J. Selfridge and R. Crandall we learned that B. Mares discovered, and proved that $\mu < \pi/4$.

and

$$\chi_a(x) := \begin{cases} 1 & \text{if } 0 \le x < a \\ \frac{1}{2} & \text{if } x = a \\ 0 & \text{if } x > a. \end{cases}$$

We first state some standard results about the Fourier cosine transform (FCT) which may be found in texts such as [4, ch. 13]. The FCT of a given function $f \in L_2(0,\infty)$ is defined to be the function F that is the L_2-limit as $y \to \infty$ of

$$c_y(x) := \sqrt{\frac{2}{\pi}} \int_0^y f(t)\cos(xt)\,dt, \text{ i.e. } \int_0^\infty |c_y(x) - F(x)|^2\,dx \to 0 \text{ as } y \to \infty.$$

This function F exists, is unique apart from sets of zero Lebesgue measure, $F \in L_2(0,\infty)$ and f is the FCT of F. In addition, if f is continuous on $[0,\alpha]$ for some $\alpha > 0$ and $F \in L_1(0,\infty)$, then

$$\sqrt{\frac{2}{\pi}} \int_0^\infty F(x)\cos(xt)\,dx = f(t) \text{ for } 0 \le t \le \alpha,$$

since the left-hand term is also continuous on $[0,\alpha]$ by dominated convergence. Note that, for $a > 0$, the FCT of χ_a is $a\sqrt{\frac{2}{\pi}}\mathrm{sinc}(ax)$, so that the FCT of $a\sqrt{\frac{2}{\pi}}\mathrm{sinc}(ax)$ is χ_a. Note also that if F_1, F_2 are FCTs of functions $f_1, f_2 \in L_2(0,\infty)$, then F_1F_2 is the FCT of $\sqrt{\frac{2}{\pi}} f_1 * f_2$, where

$$f_1 * f_2(x) := \int_0^x f_1(x-t) f_2(t)\,dt \text{ for } x \ge 0.$$

In addition, we have the following version of Parseval's theorem:

$$\int_0^\infty f_1(x) f_2(x)\,dx = \int_0^\infty F_1(x) F_2(x)\,dx,$$

provided at least one of the functions f_1, f_2 is real.

We are now in a position to prove:

Theorem 1. *Suppose that $\{a_n\}$ is a sequence of positive numbers. Let $s_n := \sum_{k=2}^n a_k$, and $\tau_n := \int_0^\infty \prod_{k=1}^n \mathrm{sinc}(a_k x)\,dx$.*

(i) *Then*

$$0 < \tau_n \le \frac{1}{a_1}\frac{\pi}{2},$$

with equality if $n = 1$, or if $a_1 \ge s_n$ when $n \ge 2$.

(ii) *If* $a_1 < s_{n_0}$ *with* $n_0 \geq 2$, *then*

$$0 < \tau_{n+1} < \tau_n < \tau_{n_0} < \frac{1}{a_1}\frac{\pi}{2} \text{ for } n > n_0.$$

(iii) *If* $a_1 < s_{n_0}$ *with* $n_0 \geq 2$, *and* $\sum_{k=1}^{\infty} a_k^2 < \infty$, *then*

$$\tau_n > \int_0^\infty \prod_{k=1}^{\infty} \operatorname{sinc}^2(a_k x)\, dx > 0 \text{ for } n > n_0.$$

Proof. Part (i). That $\tau_1 = \dfrac{1}{a_1}\dfrac{\pi}{2}$ is a standard result (proven e.g., by contour integration in [1, p. 157] and by Fourier analysis in [3, p. 563]) with the integral in question being improper (i.e. not absolutely convergent—the integrals in the other cases are absolutely convergent). Assume therefore that $n \geq 2$, and let

$$F_1 := \frac{1}{a_1}\sqrt{\frac{\pi}{2}}\chi_{a_1}, \quad F_n := \left(\sqrt{\frac{2}{\pi}}\right)^{n-2} f_2 * f_3 * \cdots * f_n, \text{ where } f_n := \frac{1}{a_n}\sqrt{\frac{\pi}{2}}\chi_{a_n}.$$

Then it is readily verified by induction that F_n vanishes on (s_n, ∞) and is positive and absolutely continuous on $(0, s_n)$. Also

$$F_n \text{ is the FCT of } \sigma_n(x) := \prod_{k=2}^{n} \operatorname{sinc}(a_k x), \text{ and } \sigma_n \text{ is the FCT of } F_n.$$

Thus, all our functions and transforms are in $L_1(0,\infty) \cap L_2(0,\infty)$. Hence, by the above version of Parseval's theorem,

$$\tau_n = \int_0^\infty F_n(x) F_1(x)\, dx = \frac{1}{a_1}\sqrt{\frac{\pi}{2}} \int_0^{\min(s_n, a_1)} F_n(x)\, dx. \tag{2}$$

When $a_1 \geq s_n$, the final term is equal to $\dfrac{1}{a_1}\sqrt{\dfrac{\pi}{2}}\sqrt{\dfrac{\pi}{2}}\sigma_n(0) = \dfrac{1}{a_1}\dfrac{\pi}{2}$ since $\sigma_n(x)$ is continuous on $[0,\infty)$; and when $a_1 < s_n$, the term is positive and less than $\dfrac{1}{a_1}\dfrac{\pi}{2}$ since $F_n(x)$ is positive and continuous for $0 < x < s_n$. This establishes part (i).

Part (ii). Observe that $F_{n+1} = \sqrt{\dfrac{2}{\pi}} F_n * f_{n+1}$, and hence that, for $y > y_1 > 0$,

$$\begin{aligned}\int_0^y F_{n+1}(x)\, dx &= \sqrt{\frac{2}{\pi}} \int_0^y dx \int_0^x F_n(t) f_{n+1}(x-t)\, dt \\ &= \sqrt{\frac{2}{\pi}} \int_0^y F_n(t)\, dt \int_t^y f_{n+1}(x-t)\, dx = I_1 + I_2,\end{aligned}$$

where

$$I_1 := \sqrt{\frac{2}{\pi}} \int_0^{y_1} F_n(t)\,dt \int_0^{y-t} f_{n+1}(u)\,du \text{ and}$$
$$I_2 := \sqrt{\frac{2}{\pi}} \int_{y_1}^{y} F_n(t)\,dt \int_0^{y-t} f_{n+1}(u)\,du.$$

Suppose now that $0 < y \le s_n$ and $0 < y - y_1 < \frac{1}{2}a_{n+1}$. Since

$$\int_0^v f_{n+1}(u)\,du < \frac{1}{2}\sqrt{\frac{\pi}{2}} \text{ when } 0 < v < \frac{1}{2}a_{n+1}, \text{ and}$$
$$\int_0^v f_{n+1}(u)\,du \le \sqrt{\frac{\pi}{2}} \text{ when } 0 < v,$$

we see that

$$I_1 \le \int_0^{y_1} F_n(t)\,dt \text{ and } 0 < I_2 \le \frac{1}{2}\int_{y_1}^{y} F_n(t)\,dt,$$

and hence that

$$\int_0^y F_{n+1}(x)\,dx < \int_0^y F_n(x)\,dx \text{ when } 0 < y < s_n. \tag{3}$$

It follows from (2), and (3) with $y = a_1$, that $0 < \tau_{n+1} < \tau_n$ whenever $a_1 < s_n$, and this establishes part (ii).

Part (iii). Let $\sigma^2(x) := \lim_{n\to\infty} \sigma_n^2(x) = \prod_{k=1}^{\infty} \operatorname{sinc}^2(a_k x)$ for $x > 0$. Observe that the limit exists since $0 \le \operatorname{sinc}^2(a_k x) < 1$, and that there is a set A differing from $(0,\infty)$ by a countable set such that $0 < \operatorname{sinc}^2(a_k x) < 1$ whenever $x \in A$ and $k = 1, 2, \ldots$. Now

$$\operatorname{sinc}^2(a_k x) = 1 - \delta_k, \text{ where } 0 \le \frac{\delta_k}{a_k^2} \to \frac{x^2}{3} \text{ as } k \to \infty,$$

so that $\sum_{k=1}^{\infty} \delta_k < \infty$, and hence, by standard theory of infinite products, $\sigma(x) > 0$ for $x \in A$. It follows that, for $n \ge n_0$,

$$\tau_n > \int_0^\infty \sigma_n^2(x)\,dx \ge \int_0^\infty \sigma^2(x)\,dx > 0,$$

by part (ii). □

Observe that applying this result to different permutations of the parameters exposes different inequalities. Also, part (iii) and dominated convergence imply that, subject to the hypotheses of part (iii),

$$\int_0^\infty \prod_{k=1}^{\infty} \operatorname{sinc}(a_k x)\,dx \ge \int_0^\infty \prod_{k=1}^{\infty} \operatorname{sinc}^2(a_k x)\,dx.$$

3. Some elementary identities. In this section we prove some identities involving products of sines and cosines by straightforward methods not involving Fourier transform theory. We adopt the usual convention that empty sums have the value 0 and empty products have the value 1.

Theorem 2. *Let $a_1, a_2, \ldots, a_n$, be given complex numbers.*

(i) *Then*

$$\prod_{k=1}^{n} \sin(a_k x) = \frac{1}{2^{n-1}} \sum_{k=1}^{2^{n-1}} \epsilon_k \cos\left(b_k x - \frac{\pi}{2} n\right),$$

where

$$b_k = \sum_{j=1}^{n} \gamma_j a_j, \gamma_1 = 1, \gamma_j = \pm 1, \epsilon_k = \prod_{k=1}^{n} \gamma_k = \pm 1,$$

and

$$\sum_{k=1}^{2^{n-1}} \epsilon_k b_k^r = \begin{cases} 0, & \textit{for } r = 1, 2, \ldots, n-2, \\ 2^{n-1}(n-1)! \prod_{k=2}^{n} a_k, & \textit{for } r = n-1. \end{cases}$$

(ii) *If the a_k's are real, then*

$$\int_0^\infty \prod_{k=1}^{n} \frac{\sin(a_k x)}{x}\, dx = \frac{\pi}{2} \frac{1}{2^{n-1}(n-1)!} \sum_{k=1}^{2^{n-1}} \epsilon_k b_k^{n-1} \operatorname{sign}(b_k).$$

If, in addition,

$$a_1 \geq \sum_{k=2}^{n} |a_k|,$$

then

$$\int_0^\infty \prod_{k=1}^{n} \frac{\sin(a_k x)}{x}\, dx = \frac{\pi}{2} \prod_{k=2}^{n} a_k.$$

Proof. We prove part (i) by induction, observing that it is true for the case $n = 1$. Suppose that (i) holds for a certain positive integer n, and that a_{n+1} is an arbitrary complex number. Then

$$\begin{aligned} 2^n \prod_{k=1}^{n+1} \sin(a_k x) &= 2 \sum_{k=1}^{2^{n-1}} \epsilon_k \cos\left(b_k x - \frac{\pi}{2} n\right) \sin(a_{n+1} x) \\ &= \sum_{k=1}^{2^{n-1}} \epsilon_k \left\{ \cos\left((b_k + a_{n+1})x - \frac{\pi}{2}(n+1)\right) - \cos\left((b_k - a_{n+1})x - \frac{\pi}{2}(n+1)\right) \right\} \\ &= \sum_{k=1}^{2^n} \epsilon_k' \cos\left(b_k' x - \frac{\pi}{2}(n+1)\right), \end{aligned}$$

where, for $k = 1, 2, \ldots, 2^{n-1}$,

$$\epsilon'_k := \epsilon_k,\ \epsilon'_{k+2^{n-1}} := -\epsilon_k,\ b'_k := b_k + a_{n+1},\ b'_{k+2^{n-1}} := b_k - a_{n+1}.$$

Hence, for $r = 1, 2, \ldots, n$,

$$\begin{aligned}\sum_{k=1}^{2^n} \epsilon'_k (b'_k)^r &= \sum_{k=1}^{2^{n-1}} \epsilon_k \left\{(b_k + a_{n+1})^r - (b_k - a_{n+1})^r\right\} \\ &= \sum_{k=1}^{2^{n-1}} \epsilon_k \sum_{j=0}^{r} \binom{r}{j} \left\{1 - (-1)^{r-j}\right\} b_k^j a_{n+1}^{r-j} \\ &= \sum_{j=0}^{r-1} \binom{r}{j} \left\{1 - (-1)^{r-j}\right\} a_{n+1}^{r-j} \sum_{k=1}^{2^{N-1}} \epsilon_k b_k^j.\end{aligned}$$

By the inductive hypothesis this is 0 for $r = 1, 2, \ldots, n-1$, and for $r = n$ it is equal to

$$2na_{n+1} \sum_{k=1}^{2^{n-1}} \epsilon_k b_k^{n-1} = 2^n n! \prod_{k=2}^{n+1} a_k$$

as desired. Part (i) of the theorem is thus established by induction with the value of ϵ_k as stated.

To prove part (ii) of the theorem, observe that

$$\int_0^\infty \prod_{k=1}^{n} \frac{\sin(a_k x)}{x}\, dx = \frac{1}{2^{n-1}} \int_0^\infty x^{-n} C_n(x)\, dx. \tag{4}$$

where $C_n(x) := \sum_{k=1}^{2^{n-1}} \epsilon_k \cos\left(b_k x - \frac{\pi}{2} n\right)$. Because $C_n(x)$ is an entire function, bounded for all real x, with a zero of order n at $x = 0$, we can integrate the right-hand side of (4) by parts $n-1$ times to get

$$\begin{aligned}\int_0^\infty \prod_{k=1}^{n} \frac{\sin(a_k x)}{x}\, dx &= \frac{1}{2^{n-1}(n-1)!} \int_0^\infty \frac{dx}{x} \sum_{k=1}^{2^{n-1}} \epsilon_k b_k^{n-1} \sin(b_k x) \\ &= \frac{1}{2^{n-1}(n-1)!} \sum_{k=1}^{2^{n-1}} \epsilon_k b_k^{n-1} \int_0^\infty \frac{\sin(b_k x)}{x}\, dx \\ &= \frac{\pi}{2} \frac{1}{2^{n-1}(n-1)!} \sum_{k=1}^{2^{n-1}} \epsilon_k b_k^{n-1} \operatorname{sign}(b_k).\end{aligned}$$

Since the additional hypothesis implies that $b_k \geq 0$ for $k = 1, 2, \ldots, 2^{n-1}$, the final formula in the theorem follows from part (i). $\square$

Corollary 1. *If $2a_k \geq a_n > 0$ for $k = 1, 2, \ldots, n-1$ and*

$$\sum_{k=2}^{n} a_k > a_1 \geq \sum_{k=2}^{n-1} a_k,$$

then

$$\int_0^\infty \prod_{k=1}^{r} \frac{\sin(a_k x)}{x}\, dx = \frac{\pi}{2} \prod_{k=2}^{r} a_k \text{ for } r = 1, 2, \ldots, n-1,$$

while

$$\int_0^\infty \prod_{k=1}^{n} \frac{\sin(a_k x)}{x}\, dx = \frac{\pi}{2} \left\{ \prod_{k=2}^{n} a_k - \frac{(a_2 + a_3 + \cdots + a_n - a_1)^{n-1}}{2^{n-2}(n-1)!} \right\}.$$

Proof. Observe that

$$b_{2^{n-1}} := a_1 - a_2 - \cdots - a_n < 0 \text{ so that } \epsilon_{2^{n-1}} = (-1)^{n-1},$$

and that all other $b_k{}'s$ are non-negative. It follows that

$$\begin{aligned}
\int_0^\infty \prod_{k=1}^{n} \frac{\sin(a_k x)}{x}\, dx &= \frac{\pi}{2} \frac{1}{2^{n-1}(n-1)!} \sum_{k=1}^{2^{n-1}} \epsilon_k b_k^{n-1} \operatorname{sign}(b_k) \\
&= \frac{\pi}{2} \frac{1}{2^{n-1}(n-1)!} \left(\sum_{k=1}^{2^{n-1}} \epsilon_k b_k^{n-1} + \epsilon_{2^{n-1}} b_{2^{n-1}}^{n-1} \big(\operatorname{sign}(b_{2^{n-1}}) - 1\big) \right) \\
&= \frac{\pi}{2} \left\{ \prod_{k=2}^{n} a_k - \frac{2(-b_{2^{n-1}})^{n-1}}{2^{n-1}(n-1)!} \right\},
\end{aligned}$$

as desired. □

Remarks 1. (a) If all the $a_k{}'s$ are real and nonzero, then, by Theorem 2(ii),

$$\begin{aligned}
\tau_n := \int_0^\infty \prod_{k=1}^{n} \operatorname{sinc}(a_k x)\, dx &= \frac{\pi}{2} \frac{1}{2^{n-1}(n-1)! a_1 a_2 \cdots a_n} \sum_{k=1}^{2^{n-1}} \epsilon_k b_k^{n-1} \operatorname{sign}(b_k) \\
&= \frac{\pi}{2} \frac{1}{2^{n-1}(n-1)! a_1 a_2 \cdots a_n} \left(\sum_{k=1}^{2^{n-1}} \epsilon_k b_k^{n-1} + \sum_{b_k < 0} \epsilon_k b_k^{n-1} \big(\operatorname{sign}(b_k) - 1\big) \right) \\
&= \frac{\pi}{2a_1} \left\{ 1 - \frac{2}{2^{n-1}(n-1)! a_2 a_3 \cdots a_n} \sum_{b_k < 0} \epsilon_k b_k^{n-1} \right\}.
\end{aligned}$$

(b) Suppose further that the $a_k's$ are positive. Consider the polyhedron $P(n)$ given by

$$P(a_1, a_2, \cdots, a_n) := \{(x_2, x_3, \cdots, x_n) : \sum_{k=2}^{n} a_k \le a_1, 0 \le x_k \le a_k, 2 \le k \le n\}.$$

If we return to equation (2) we may observe that

$$\tau_n = \frac{\pi}{2a_1}\frac{1}{a_2\, a_3 \cdots a_n}\int_0^{\min(s_n, a_1)} \chi_{a_2} * \chi_{a_3} * \cdots * \chi_{a_n}\, dx = \frac{\pi}{2a_1}\frac{Vol(P(n))}{a_2\, a_3 \cdots a_n}.$$

Thus, in (a) we have evaluated the volume of $P(n)$. Moreover, we now explain the behaviour of τ_n when we note that the value drops precisely when the constraint $\sum_{k=2}^{n} x_k \le a_1$ becomes active and bites into the hypercube $\{(x_2, x_3, \cdots, x_n) : 0 \le x_k \le a_k, 2 \le k \le n\}$.

(c) Consider now the special case

$$\tau_n = \mu_n := \int_0^{\infty} \operatorname{sinc}^n(x)\, dx.$$

In this case we have $a_k = 1$ for all k, and it is straightforward to verify that

$$\sum_{b_k<0} \epsilon_k b_k^{n-1} = \sum_{1\le r\le \frac{n}{2}} (-1)^{r+1}\binom{n-1}{r-1}(n-2r)^{n-1},$$

and hence that

$$\begin{aligned}\mu_n :=& \frac{\pi}{2}\left\{1 - \frac{2}{2^{n-1}(n-1)!}\sum_{1\le r\le \frac{n}{2}} (-1)^{r+1}\binom{n-1}{r-1}(n-2r)^{n-1}\right\}\\ =& \frac{\pi}{2}\left\{1 + \frac{1}{2^{n-2}}\sum_{1\le r\le \frac{n}{2}} \frac{(-1)^r}{(r-1)!}\frac{(n-2r)^{n-1}}{(n-r)!}\right\}.\end{aligned}$$

□

The next theorem extends Theorem 2 by adjoining cosines to the product of sines.

Theorem 3. *Let $a_1, a_2, \ldots, a_{n+m}$, be given complex numbers, m, n being non-negative integers with $n \ge 1$.*

(i) *Then*

$$\left(\prod_{k=1}^{n}\sin(a_k x)\right)\left(\prod_{k=n+1}^{n+m}\cos(a_k x)\right) = \frac{1}{2^{n+m-1}}\sum_{k=1}^{2^{n+m-1}} \epsilon_k \cos\left(b_k x - \frac{\pi}{2}n\right),$$

where

$$b_k = \sum_{j=1}^{n+m} \gamma_j a_j, \gamma_1 = 1, \gamma_j = \pm 1, \epsilon_k = \prod_{k=1}^{n+m} \gamma_k = \pm 1,$$

and

$$\sum_{k=1}^{2^{n+m-1}} \epsilon_k b_k^r = \begin{cases} 0, & \textit{for } r = 1, 2, \ldots, n-2, \\ 2^{n+m-1}(n-1)! \prod_{k=2}^{n} a_k, & \textit{for } r = n-1. \end{cases}$$

(ii) *If the a_k's are real, then*

$$\int_0^\infty \left(\prod_{k=1}^{n} \frac{\sin(a_k x)}{x} \right) \left(\prod_{k=n+1}^{n+m} \cos(a_k x) \right) dx$$

$$= \frac{\pi}{2} \frac{1}{2^{n+m-1}(n-1)!} \sum_{k=1}^{2^{n+m-1}} \epsilon_k b_k^{n-1} \operatorname{sign}(b_k) \; .$$

If, in addition,

$$a_1 \geq \sum_{k=2}^{n+m} |a_k|,$$

then

$$\int_0^\infty \left(\prod_{k=1}^{n} \frac{\sin(a_k x)}{x} \right) \left(\prod_{k=n+1}^{n+m} \cos(a_k x) \right) dx = \frac{\pi}{2} \prod_{k=2}^{n} a_k.$$

Proof. By Theorem 2 we have that

$$\prod_{k=1}^{n+m} \sin(a_k x) = \frac{1}{2^{n+m-1}} \sum_{k=1}^{2^{n+m-1}} \epsilon'_k \cos\left(b_k x - \frac{\pi}{2}(n+m) \right),$$

where

$$b_k = \sum_{j=1}^{n+m} \gamma_j a_j, \gamma_1 = 1, \gamma_j = \pm 1, \epsilon'_k = \prod_{k=1}^{n+m} \gamma_k = \pm 1,$$

and

$$\sum_{k=1}^{2^{n+m-1}} \epsilon'_k b_k^r = \begin{cases} 0, & \text{for } r = 1, 2, \ldots, n+m-2, \\ 2^{n+m-1}(n+m-1)! \prod_{k=2}^{n+m} a_k, & \text{for } r = n+m-1. \end{cases}$$

Differentiating these expressions partially with respect to $a_{n+1}, a_{n+2}, \ldots, a_{n+m}$ yields part (i) of Theorem 3 with $\epsilon_k = \gamma^m \epsilon'_k$. To deal with part (ii) of Theorem 3 we observe that, by Theorem 2, if the a_k's are real, then

$$\int_0^\infty \prod_{k=1}^{n+m} \frac{\sin(a_k x)}{x} dx = \frac{\pi}{2} \frac{1}{2^{n+m-1}(n+m-1)!} \sum_{k=1}^{2^{n+m-1}} \epsilon'_k b_k^{n-1} \operatorname{sign}(b_k).$$

Differentiating partially with respect to $a_{n+1}, a_{n+2}, \ldots, a_{n+m}$, we get

$$\int_0^\infty \left(\prod_{k=1}^{n} \frac{\sin(a_k x)}{x}\right)\left(\prod_{k=n+1}^{n+m} \cos(a_k x)\right) dx$$
$$= \frac{\pi}{2} \frac{1}{2^{n+m-1}(n-1)!} \sum_{k=1}^{2^{n+m-1}} \epsilon_k b_k^{n-1} \operatorname{sign}(b_k).$$

If, in addition,

$$a_1 \geq \sum_{k=2}^{n+m} |a_k|,$$

then, by Theorem 2,

$$\int_0^\infty \prod_{k=1}^{n+m} \frac{\sin(a_k x)}{x}\, dx = \frac{\pi}{2} \prod_{k=2}^{n+m} a_k.$$

Differentiating partially with respect to $a_{n+1}, a_{n+2}, \ldots, a_{n+m}$, we get

$$\int_0^\infty \left(\prod_{k=1}^{n} \frac{\sin(a_k x)}{x}\right)\left(\prod_{k=n+1}^{n+m} \cos(a_k x)\right) dx = \frac{\pi}{2} \prod_{k=2}^{n} a_k. \qquad \square$$

Corollary 2. *If* $2a_k \geq a_{n+m} > 0$ *for* $k = 1, 2, \ldots, n+m-1$ *and*

$$\sum_{k=2}^{n+m} a_k > a_1 \geq \sum_{k=2}^{n+m-1} a_k,$$

then

$$\int_0^\infty \left(\prod_{k=1}^{r} \frac{\sin(a_k x)}{x}\right)\left(\prod_{k=r+1}^{r+m} \cos(a_k x)\right) dx = \frac{\pi}{2} \prod_{k=2}^{n} a_k \ \textit{for}\ r = 1, 2, \ldots, n-1,$$

while

$$\int_0^\infty \left(\prod_{k=1}^{n} \frac{\sin(a_k x)}{x}\right)\left(\prod_{k=n+1}^{n+m} \cos(a_k x)\right) dx$$
$$= \frac{\pi}{2} \left\{ \prod_{k=2}^{n} a_k - \frac{(a_2 + a_3 + \cdots + a_{n+m+1} - a_1)^{n-1}}{2^{n+m-2}(n-1)!} \right\}.$$

Proof. The first part follows immediately from Theorem 3, and the second part can be derived from Corollary 1 with $n+m$ in place of n by differentiating partially with respect to $a_{n+1}, a_{n+2}, \ldots, a_{n+m}$, as above. $\square$

4. An alternative proof. The next theorem is a restatement of the last part of Theorem 3 restricted to real numbers. It appears as an example without proof in [5, p. 122] where it is ascribed to Carl Störmer [2]. Störmer's article does not contain the integral in question, but his proof for the series identity

$$\sum_{r=1}^{\infty}(-1)^{r+1}\left(\prod_{k=1}^{n}\frac{\sin(ra_k)}{r}\right)\left(\prod_{j=1}^{m}\cos(rc_j)\right)=\frac{1}{2}\prod_{k=1}^{n}a_k,$$

$$\text{provided } \sum_{k=1}^{n}|a_k|+\sum_{j=1}^{m}|c_j|<\pi,$$

is readily adapted to yield a proof of the theorem which is radically different from the proof of Theorem 3.

Theorem 4. *If* $a, a_1, a_2, \ldots, a_n, c_1, c_2, \ldots, c_m$, *are real numbers with*

$$a>\sum_{k=1}^{n}|a_k|+\sum_{j=1}^{m}|c_j|,$$

then

$$\int_0^{\infty}\left(\prod_{k=1}^{n}\frac{\sin(a_kx)}{x}\right)\left(\prod_{j=1}^{m}\cos(c_jx)\right)\frac{\sin(ax)}{x}\,dx=\frac{\pi}{2}\prod_{k=1}^{n}a_k. \tag{5}$$

Proof. We prove the theorem by induction. Applying as before the convention that empty sums have the value 0 and empty products have the value 1, we observe that formula (5) for the case $n = m = 0$ reduces to the standard result

$$\int_0^{\infty}\frac{\sin(ax)}{x}\,dx=\frac{\pi}{2} \text{ when } a>0.$$

Formula (5) also holds for the case $n = 1, m = 0$, by the case $n = 2$ of Theorem 1 (which can easily be proved directly).

Assume that the theorem holds for certain integers $n \geq 1$ and $m \geq 0$. First suppose that

$$a>\sum_{k=1}^{n}|a_k|+\sum_{j=1}^{m+1}|c_j|.$$

Then

$$a>|a_1\pm c_{m+1}|+\sum_{k=2}^{n}|a_k|+\sum_{j=1}^{m}|c_j|,$$

and hence

$$\int_0^\infty \frac{\sin(a_1 \pm c_{m+1})}{x} \left(\prod_{k=2}^{n} \frac{\sin(a_k x)}{x}\right) \left(\prod_{j=1}^{m} \cos(c_j x)\right) \frac{\sin(ax)}{x}\, dx = \frac{\pi}{2}(a_1 \pm c_{m+1}) \prod_{k=2}^{n} a_k. \tag{6}$$

Adding the two identities in (6), we immediately obtain

$$\int_0^\infty \left(\prod_{k=1}^{n} \frac{\sin(a_k x)}{x}\right) \left(\prod_{j=1}^{m+1} \cos(c_j x)\right) \frac{\sin(ax)}{x}\, dx = \frac{\pi}{2} \prod_{k=1}^{n} a_k. \tag{7}$$

Next suppose that

$$a > \sum_{k=1}^{n+1} |a_k| + \sum_{j=1}^{m} |c_j|,$$

and let t lie between 0 and a_{n+1}. Then, by (7), we have

$$\int_0^\infty \left(\prod_{k=1}^{n} \frac{\sin(a_k x)}{x}\right) \left(\prod_{j=1}^{m} \cos(c_j x)\right) \cos(tx)\frac{\sin(ax)}{x}\, dx = \frac{\pi}{2} \prod_{k=1}^{n} a_k. \tag{8}$$

Now integrate (8) with respect to t from 0 to a_{n+1} to get

$$\int_0^\infty \left(\prod_{k=1}^{n+1} \frac{\sin(a_k x)}{x}\right) \left(\prod_{j=1}^{m} \cos(c_j x)\right) \frac{\sin(ax)}{x}\, dx = \frac{\pi}{2} \prod_{k=1}^{n+1} a_k. \tag{9}$$

Identities (7) and (9) show that if the theorem holds for a pair of integers n, m with $n \geq 1, m \geq 0$, then it also holds for the pairs $n, m+1$ and $n+1, m$. Since it holds for $n = 1, m = 0$, the proof is completed by induction. $\square$

Remarks 2. Parts of our previous theorems do, of course, overlap with Theorem 4, but this latter theorem does not deal with cases where the identity in (4) fails, whereas the other theorems do. Thus, for example,

$$\int_0^\infty \operatorname{sinc}(x)\, dx = \frac{\pi}{2},$$

$$\int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{3}\right)\, dx = \frac{\pi}{2},$$

$$\cdots$$

$$\int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{3}\right)\cdots\operatorname{sinc}\left(\frac{x}{13}\right)\, dx = \frac{\pi}{2},$$

yet

$$\int_0^\infty \operatorname{sinc}(x)\operatorname{sinc}\left(\frac{x}{3}\right)\cdots\operatorname{sinc}\left(\frac{x}{15}\right)\,dx = \frac{467807924713440738696537864469}{935615849440640907310521750000}\pi, \tag{10}$$

and this fraction in (10), in accord with Corollary 1, is approximately equal to 0.499999999992646. When this fact was recently verified by a researcher using a computer algebra package, he concluded that there must be a "bug" in the software. Not so. In the above example, $\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{13} < 1$, but with the addition of $\frac{1}{15}$, the sum exceeds 1 and the identity no longer holds. This is a somewhat cautionary example for too enthusiastically inferring patterns from symbolic or numerical computation. □

5. An infinite product of cosines. We return to the integral, which we denote by μ, in (1). Let

$$C(x) := \prod_{n=1}^{\infty} \cos\left(\frac{x}{n}\right).$$

Recall Vieta's formula [3, p. 419] in the form

$$\operatorname{sinc}(x) = \prod_{n=0}^{\infty} \cos\left(\frac{x}{2^n}\right),$$

and relatedly the product expansion

$$\operatorname{sinc}(\pi x) = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right).$$

We may thus re-express C as the absolutely convergent product:

$$C(x) = \prod_{n=0}^{\infty} \operatorname{sinc}\left(\frac{2x}{2n+1}\right) \tag{11}$$

and apply Theorem 1 to obtain

$$0 < \mu = \int_0^\infty C(x)\,dx = \lim_{N\to\infty} \int_0^\infty \prod_{k=1}^{N} \operatorname{sinc}\left(\frac{2x}{2k-1}\right)\,dx < \frac{\pi}{4}.$$

These sinc integrals are essentially those of the previous Remarks. Note that all parts of Theorem 1 apply since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} < \infty = \sum_{k=1}^{\infty} \frac{1}{2k-1}$.

We observe that Theorem 1 allows for reasonable lower bounds on μ. Indeed, as $\cos^2 x > 1-x^2 > 0$ for $0 < x < 1$, we see — using the product form for sinc — that $C^2(x) > \operatorname{sinc}(\pi x)$ on the same range. Hence, by Theorem 1(iii),

$$\frac{\pi}{4} > \mu > \int_0^\infty C^2(x)\,dx > \frac{1}{\pi}\int_0^\pi \operatorname{sinc}(x)\,dx \approx .5894898721.$$

We could produce a better lower bound, and indeed lower bounds for our more general sinc integrals in the same way.

In fact

$$\int_0^\infty C(x)\,dx \approx 0.785380557298632873492583011467332524761$$

while $\frac{\pi}{4} \approx .785398$ only differs in the fifth significant place. We note that high precision numerical evaluation of these highly oscillatory integrals is by no means straightforward.

We finish by recording without details that (11) allows us to obtain the Taylor series around 0 for $\log C$. It is

$$\log C(x) = -\sum_{k=1}^\infty \frac{4^k-1}{k}\frac{\zeta^2(2k)}{\pi^{2k}}x^{2k},$$

with radius of convergence $\pi/2$. This in turn shows that the coefficient of x^{2n} in the Taylor series for C, say c_n, is a rational multiple of π^{2k} and is explicitly given by the recursion

$$c_n := \frac{1}{n}\sum_{k=1}^{n-1}(4^k-1)\frac{\zeta^2(2k)}{\pi^{2k}}c_{n-k}.$$

Thus

$$C(x) = 1 - \frac{1}{12}\pi^2x^2 + \frac{11}{4320}\pi^4x^4 - \frac{233}{5443200}\pi^6x^6 + \frac{1429}{3048192000}\pi^8x^8 + O\left(x^9\right).$$

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